Петак 23.04.2021.

① Нека је $a_n>0$ и $\underbrace{\frac{a_n}{a_{n+1}}=1+\frac{\alpha}{n}+o\left(\frac{1}{n}\right)}$, $n\to\infty$, $\alpha>0$. Онда $a_n\to 0$ и $a_n\searrow$ ($\Rightarrow$(Лајбниц) $\sum(-1)^n a_n$ конв.)

$$\frac{a_n}{a_{n+1}}>1\ ,\ \forall n\geq n_0 \Rightarrow a_n\searrow$$

$$n\cdot\left(\frac{a_n}{a_{n+1}}-1\right)=\alpha+o(1) \underset{n\to\infty}{} \Rightarrow \lim_{n\to\infty} n\cdot\left(\frac{a_n}{a_{n+1}}-1\right)=\alpha$$

$$(\forall\varepsilon>0)\ (\exists n_1\in\mathbb{N})\ (\forall n\geq n_1) \Rightarrow \alpha-\varepsilon<n\cdot\left(\frac{a_n}{a_{n+1}}-1\right)<\alpha+\varepsilon \quad /:n\ /+1$$

$$\boxed{\underbrace{\frac{\alpha-\varepsilon}{n}+1}_{>1}<\underbrace{\frac{a_n}{a_{n+1}}}_{>0}}<\frac{\alpha+\varepsilon}{n}+1$$

$$\frac{a_{n_1}}{a_{N+1}}=\underbrace{\frac{a_{n_1}}{a_{n_1+1}}}_{>\frac{\alpha-\varepsilon}{n_1}+1}\cdot\underbrace{\frac{a_{n_1+1}}{a_{n_1+2}}}_{>\frac{\alpha-\varepsilon}{n_1+1}+1}\cdot\ \ldots\ \cdot\underbrace{\frac{a_N}{a_{N+1}}}_{>\frac{\alpha-\varepsilon}{N+1}+1}>\left(1+\frac{\alpha-\varepsilon}{n_1}\right)\cdot\left(1+\frac{\alpha-\varepsilon}{n_1+1}\right)\cdots\left(1+\frac{\alpha-\varepsilon}{N+1}\right)\geq 1+(\alpha-\varepsilon)\left(\frac{1}{n_1}+\ldots+\frac{1}{N}\right)$$

$$\Downarrow$$

$$0<a_{N+1}<\underbrace{\frac{a_{n_1}}{1+(\alpha-\varepsilon)\underbrace{\left(\frac{1}{n_1}+\ldots+\frac{1}{N}\right)}_{\to\infty}}}_{\to 0}\quad /\lim_{N\to\infty}$$

[$(1+x_1)(1+x_2)\cdots(1+x_n)\geq 1+x_1+\ldots+x_n$

за $x_1,\ldots,x_n>0$

уопштење од: $(1+x)^n\geq 1+nx$]

$$0\leq\lim_{N\to\infty}a_N\leq 0\Rightarrow\lim_{N\to\infty}a_N=0.$$

② $\sum\binom{\alpha}{n}$ испитати конв. $\alpha\in\mathbb{R}$

[ако $\alpha\in\mathbb{N}$, $a_n=0$ од неког] $\Rightarrow$ конв.

$$a_n=\binom{\alpha}{n}=\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}\qquad \binom{\alpha}{0}=1$$

$$\frac{a_n}{a_{n+1}}=\frac{\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}}{\frac{\alpha(\alpha-1)\cdots(\alpha-n)}{(n+1)!}}=\frac{(n+1)!}{n!}\cdot\frac{1}{\alpha-n}=\frac{n+1}{\alpha-n}<0\ ,\ \text{за } n\geq[\alpha]+1=n_0$$

$\alpha\notin\mathbb{N}$

$\binom{\alpha}{n}=(-1)^n\cdot b_n$, b_n – сталног знака (рецимо $b_n>0$)

[$(-1)^{n+1}b_n$]

$$\frac{b_n}{b_{n+1}} = \frac{|a_n|}{|a_{n+1}|} = \left|\frac{n+1}{\alpha-n}\right| = \frac{n+1}{n-\alpha} = \left(1+\frac{1}{n}\right)\left(1-\frac{\alpha}{n}\right)^{-1} = \left(1+\frac{1}{n}\right)\left(1+\frac{\alpha}{n}+o\left(\frac{1}{n}\right)\right) = 1+\frac{\alpha+1}{n}+o\left(\frac{1}{n}\right)$$

- $\alpha+1>0$, $\alpha>-1 \overset{①}{\Rightarrow} b_n \searrow 0 \Rightarrow \sum a_n$ конв.
- $\alpha+1\leq 0$, $\alpha\leq -1 \Rightarrow \dfrac{|a_n|}{|a_{n+1}|}\leq 1 \Rightarrow |a_n|\leq|a_{n+1}|$ (од неког n)

  $|a_n|\nearrow$ и $|a_n|>0 \Rightarrow |a_n|\not\to 0 \Rightarrow$ див.

додати: $\sum (-1)^n \binom{\alpha}{n}$ конв. $\Leftrightarrow \sum \left|\binom{\alpha}{n}\right| < \infty$

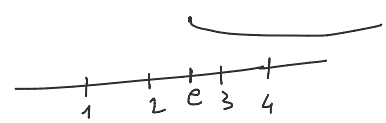
$(-1)^n\binom{\alpha}{n}>0$, $\forall n\geq n_0$

③ Ако $\sum a_n$ конв. $\Rightarrow \sum \sqrt[n]{n}\cdot a_n$ конв.

(A1) $\sum a_n$ конв.

(A2) $b_n=\sqrt[n]{n}$ монотон и $\exists \lim b_n$?

$$f(x)=x^{1/x}$$
$$f'(x)=\left(e^{\frac{\log x}{x}}\right)' = e^{\frac{\log x}{x}}\cdot\frac{\frac{1}{x}\cdot x-\log x\cdot 1}{x^2} = x^{1/x}\cdot\frac{1-\log x}{x^2} < 0 \text{ , за } x>e$$



$\Rightarrow b_n \downarrow$ за $n\geq 3$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\sqrt[n]{n} = 1$$

По Абелу $\Rightarrow \sum \sqrt[n]{n}\cdot a_n$ конв.

④ a_n је низ тд. $a_{n+3}=a_n$. $\sum \frac{a_n}{n}$ конв. $\Leftrightarrow a_1+a_2+a_3=0$.

$a_1 = a_4 = a_7$
$a_2 = a_5 = a_8$ ....
$a_3 = a_6 = a_9$

$\boxed{\Rightarrow}$ $\sum \frac{a_n}{n}$ конв. $\Rightarrow \exists \lim S_N \Rightarrow \exists \lim S_{3N}$

$a_1+a_2+a_3=A$

$$S_{3N} = \left(\frac{a_1}{1}+\frac{a_2}{2}+\frac{a_3}{3}\right)+\left(\frac{a_1}{4}+\frac{a_2}{5}+\frac{a_3}{6}\right)+\dots+\left(\frac{a_1}{3N-2}+\frac{a_2}{3N-1}+\frac{a_3}{3N}\right) =$$

$$S_{3N} = \left(\frac{a_1}{1}+\frac{a_2}{2}+\frac{a_3}{3}\right)+\left(\frac{a_1}{4}+\frac{a_2}{5}+\frac{a_3}{6}\right)+\dots+\left(\frac{a_1}{3N-2}+\frac{a_2}{3N-1}+\frac{a_3}{3N}\right) =$$

$$= a_1\cdot\sum_{n=1}^{N}\frac{1}{3n-2} + a_2\cdot\sum_{n=1}^{N}\frac{1}{3n-1} + \underbrace{a_3}_{A-a_1-a_2}\cdot\sum_{n=1}^{N}\frac{1}{3n} =$$

$$= a_1\cdot\sum_{n=1}^{N}\left(\frac{1}{3n-2}-\frac{1}{3n}\right) + a_2\cdot\sum_{n=1}^{N}\left(\frac{1}{3n-1}-\frac{1}{3n}\right) + A\cdot\sum_{n=1}^{N}\frac{1}{3n} =$$

$$= a_1\cdot\sum_{n=1}^{N}\underbrace{\frac{2}{(3n-2)\cdot 3n}}_{\sim\frac{2}{9n^2}} + a_2\cdot\sum_{n=1}^{N}\underbrace{\frac{1}{(3n-1)\cdot 3n}}_{\sim\frac{1}{9n^2}} + \boxed{A\cdot\sum_{n=1}^{N}\frac{1}{3n}}$$

$\exists \lim$

$\lim\limits_{N\to\infty}$: $\exists \lim$ $\exists \lim$

$$\Rightarrow \exists \lim_{N\to\infty}\sum_{n=1}^{N}\frac{A}{3}\cdot\underbrace{\frac{1}{n}}_{\text{див}}$$

$$\Rightarrow \frac{A}{3}=0 \Rightarrow A=0$$

$\boxed{\Leftarrow}$ $\sum \frac{a_n}{n} = \sum a_n\cdot b_n$, $a_1+a_2+a_3=0$

$b_n = \frac{1}{n}$

(Д1) $b_n \searrow 0$ и b_n монотон

(Д2) $S_N = a_1+\dots+a_N$ огр. низ?

$$S_{3N} = a_1+\dots+a_{3N} = \underbrace{(a_1+a_2+a_3)}_{=0} + \underbrace{(a_4+a_5+a_6)}_{=0} + \dots + \underbrace{(a_{3N-2}+a_{3N-1}+a_{3N})}_{=0} = 0$$

$$S_{3N+1} = a_1+\dots+a_{3N+1} = S_{3N}+a_{3N+1} = a_{3N+1} = a_1$$

$$S_{3N+2} = S_{3N}+a_{3N+1}+a_{3N+2} = a_1+a_2$$

$|S_N| \le \max\{|a_1|, |a_1+a_2|\} = M \Rightarrow S_N$ огр. са M

$\underset{\text{Дирихле}}{\Rightarrow}$ $\sum \frac{a_n}{n}$ конв.

Степенни редове

Деф. Ред $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ се нарича степенен ред с център x_0

<u>Деф.</u> Ред $f(x) = \sum_{n=1}^{\infty} a_n (x-x_0)^n$ се назива степени ред са центром x_0

a_n – коефицијенти ($x_0, a_n \in \mathbb{R}$, може и $\mathbb{C}$)

Питање: за које x ред конвергира? (домен фје f) $\to$ увек бар за $x = x_0$

$\overset{\to D}{\text{домен}}$

$\parallel$ област конвергенције

$\boxed{T}$ Област конв. је (у $\mathbb{R}$) интервал са центром x_0 облика

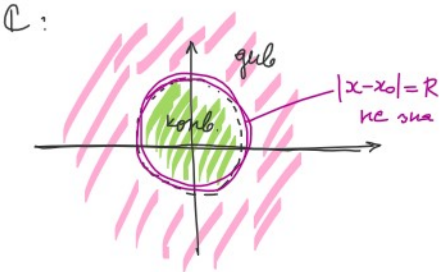
$\downarrow$ D

$(x_0 - R, x_0 + R)$, $(x_0 - R, x_0 + R]$, $[x_0 - R, x_0 + R)$, $[x_0 - R, x_0 + R]$

(или $\mathbb{R}$, $\{x_0\}$)

- ако је $|x - x_0| < R$ ред апс. конв.
- ако је $|x - x_0| > R$ ред див.
- ако је $|x - x_0| = R$ – не знамо

у $\mathbb{C}$:



у $\mathbb{R}$:

$$R = \frac{1}{\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|} = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$$

ако овај lim десно $\exists$ (у $\overline{\mathbb{R}}$)

конвенција: $\overline{\lim} = \infty \Rightarrow R = 0$ ($\{x_0\}$)

$\overline{\lim} = 0 \Rightarrow R = \infty$ ($\mathbb{R}$)

Примери: 1) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\quad x_0 = 0$, $a_n = \frac{1}{n!}$

$$R = \lim_{n\to\infty} \left| \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} \right| = \lim_{n\to\infty} (n+1) = \infty \Rightarrow D = \mathbb{R}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} : \mathbb{R} \to \mathbb{R}$$

важи: $f(x) = e^x$

2) $\sin x = \sum^{\infty} \frac{(-1)^n x^{2n+1}}{}$, $x_0 = 0$

2) $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, $x_0 = 0$

$$= x + 0\cdot x^2 - \frac{x^3}{3!} + 0\cdot x^4 + \frac{x^5}{5!} + \dots$$

$4n+1 \to +$

$4n+3 \to -$

$(-1)^{\frac{n-1}{2}}$

$$a_n = \begin{cases} 0 & , 2 \mid n \\ \frac{1}{n!}(-1)^{\frac{n-1}{2}} & , 2 \nmid n \end{cases}$$

$$\sqrt[n]{|a_n|} = \begin{cases} 0 & , 2 \mid n \\ \sqrt[n]{\frac{1}{n!}} & , 2 \nmid n \end{cases}$$

$$\overline{\lim} \sqrt[n]{|a_n|} = \max\left\{0, \lim \sqrt[n]{\frac{1}{n!}}\right\} = \lim \sqrt[n]{\frac{1}{n!}} = 0 \Rightarrow R = \frac{1}{\overline{\lim} \sqrt[n]{|a_n|}} = \infty .$$

$$n! \sim \sqrt{2n\pi}\left(\frac{n}{e}\right)^n$$

$$\sqrt[n]{\frac{1}{n!}} \sim \underbrace{\frac{e}{n}}_{\to 0} \cdot \underbrace{\frac{1}{(\sqrt{2\pi n})^{1/n}}}_{\to 1} \to 0$$

$\Rightarrow D = \mathbb{R}$

3) домашно: $\cos x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}$ $(D = \mathbb{R})$

4) $\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{x^n}{n}$

$$a_n = (-1)^{n+1} \cdot \frac{1}{n} \quad , x_0 = 0$$

$$R = \frac{1}{\overline{\lim} \sqrt[n]{|a_n|}} = \frac{1}{\overline{\lim} \sqrt[n]{\frac{1}{n}}} = \frac{1}{\frac{1}{\underbrace{\lim \sqrt[n]{n}}_{=1}}} = 1$$

$|x| < 1$, $x \in (-1, 1)$: конвергира

$|x| > 1$, $x \in (-\infty, -1) \cup (1, \infty)$: дивергира

$x = -1$ и $x = 1$? $\to$ ручно проверяваме

$x = -1$: $\sum^{\infty} (-1)^{n+1} \frac{(-1)^n}{\dots}$ $\sum^{\infty} -1 \dots$

$x=-1$: $\sum_{n=1}^{\infty} (-1)^{n+1}\cdot \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} \frac{-1}{n} = -\infty$ див.

$x=1$: $\sum_{n=1}^{\infty} (-1)^{n+1}\cdot \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ конв. (Лајбниц)

$\Big\}\ D = (-1,1]$.

5) $(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n}\cdot x^n$, $\alpha \in \mathbb{R}$

$\alpha \in \{0,1,2,\dots\}$ – посебно, $a_n = 0$, $\forall n \geq n_0$
$\Rightarrow D = \mathbb{R}$

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{n+1}{\alpha-n}\right| = \lim_{n\to\infty} \frac{n+1}{n-\alpha} = 1$$

$(-1,1)$ ✓

$|x|>1$ ×

$x=-1$: $\sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n}$ $\nearrow \alpha \geq 0$ кон. $\searrow \alpha < 0$ див.

$x=1$: $\sum_{n=0}^{\infty} \binom{\alpha}{n}$ $\nearrow \alpha \leq -1$ див. $\searrow \alpha > -1$ конв.

1° $\alpha \geq 0$, $D = [-1,1]$

2° $\alpha \in (-1,0)$, $D = (-1,1]$

3° $\alpha \leq -1$, $D = (-1,1)$

5. $\sum_{n=1}^{\infty} \frac{3^n + (-2)^n}{n} (x+1)^n$

$a_n = \frac{3^n+(-2)^n}{n}$, $x_0 = -1$

$$R = \frac{1}{\overline{\lim} \sqrt[n]{\left|\frac{3^n+(-2)^n}{n}\right|}} = \frac{1}{\overline{\lim} \frac{\sqrt[n]{3^n+(-2)^n}}{\sqrt[n]{n}}} = \frac{1}{3}$$

$$\overline{\lim} \sqrt[n]{\frac{3^n+(-2)^n}{n}} = \max\left\{ \lim \sqrt[2n]{\frac{3^{2n}+2^{2n}}{2n}},\ \lim \sqrt[2n+1]{\frac{3^{2n+1}-2^{2n+1}}{2n+1}} \right\} =$$

$$= \max\{3,3\} = 3.$$

$\lim_{n\to\infty} \sqrt[n]{a_1^n + \dots + a_m^n} = \max_{1\leq k\leq m} (a_k)$

$a_1, \dots, a_m \geq 0$

$$\sqrt[2n]{\frac{3^{2n}}{2n}} \leq \sqrt[2n]{\frac{3^{2n}+2^{2n}}{2n}} \leq \sqrt[2n]{\frac{2\cdot 3^{2n}}{2n}}$$

$\to 3$, $\to 1$, $\to 3$

$a_1, \dots, a_n \geq 0$

$$\sqrt[2n]{\frac{3^{2n}}{2n}} \leq \sqrt[2n]{\frac{3^{2n}+2^{2n}}{2n}} \leq \sqrt[2n]{\frac{2\cdot 3^{2n}}{2n}}$$

$\to 3$

$$\sqrt[2n+1]{\frac{\frac{1}{2}\cdot 3^{2n+1}}{2n+1}} \leq \sqrt[2n+1]{\frac{3^{2n+1}-2^{2n+1}}{2n+1}} \leq \sqrt[2n+1]{\frac{3^{2n+1}}{2n+1}}$$

$\to 3 \qquad \to 3 \qquad \to 3$

$\frac{1}{2}\cdot 3^{2n+1} \geq 2^{2n+1}$

$\frac{1}{2} \geq \left(\frac{2}{3}\right)^{2n+1}, \ n \geq 2$

$|x+1| < \frac{1}{3}$ konv.

$|x+1| > \frac{1}{3}$ div.

$|x+1| = \frac{1}{3} \to x = -\frac{2}{3}, \quad x = -\frac{4}{3}$

$x=-\frac{4}{3}$: $\sum \frac{3^n+(-2)^n}{n}\cdot\left(-\frac{1}{3}\right)^n = \sum\left(\underbrace{\frac{(-1)^n}{n}}_{\text{konv. po Lajbnicu}} + \underbrace{\frac{1}{n}\cdot\left(\frac{2}{3}\right)^n}\right)$ — konv.

$\frac{1}{n}\cdot\left(\frac{2}{3}\right)^n \leq \left(\frac{2}{3}\right)^n$ konv. $\frac{2}{3}<1$

$x=-\frac{2}{3}$: $\sum \frac{3^n+(-2)^n}{n}\cdot\left(\frac{1}{3}\right)^n = \sum\left(\underbrace{\frac{1}{n}}_{\text{div.}} + \underbrace{(-1)^n\cdot\frac{1}{n}\cdot\left(\frac{2}{3}\right)^n}_{\text{aps. konv.}}\right)$ — div.

$\Rightarrow D = \left[-\frac{4}{3}, -\frac{2}{3}\right).$