Понедељак 22.02.2021.

(1) $$L=\lim_{n\to\infty}\Big(\underbrace{\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\dots+\frac{1}{\sqrt{n^2+n}}}_{n \text{ чланова}}\Big)=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k}}$$

највећи ... најмањи

$$\frac{1}{\sqrt{n^2+n}}\le\frac{1}{\sqrt{n^2+k}}\le\frac{1}{\sqrt{n^2+1}},\quad \forall k\in\{1,\dots,n\}$$

$$n\cdot\frac{1}{\sqrt{n^2+n}}\le\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+n}}\le\underbrace{\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k}}}_{L}\le\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+1}}=n\cdot\frac{1}{\sqrt{n^2+1}}$$

$$\frac{n}{\sqrt{n^2+n}}\le L\le\frac{n}{\sqrt{n^2+1}}\qquad /\lim$$

$$\frac{n}{\sqrt{n^2+n}}\overset{:n}{=}\frac{1}{\sqrt{1+\frac{1}{n}}}\to\frac{1}{1}=1$$

$$\downarrow \qquad \xrightarrow{n\to\infty} \qquad \downarrow$$

$$1 \qquad\qquad 1$$

$$\Rightarrow \boxed{L=1}$$

$x_n: 1,-1,1,-1,\dots$

$\{x_n \mid n\in\mathbb{N}\}=\{1,-1\}.$

деф. $\{x_n \mid n\in\mathbb{N}\}$

(2) Наћи инфимум и супремум скупа вредности низа $(x_n)_{n\in\mathbb{N}}$, као и његове Т.Н. и $\underline{\lim}$, $\overline{\lim}$.

а) $x_n=(-1)^{n-1}\left(2+\frac{3}{n}\right)$

б) $x_n=1+\frac{n}{n+1}\cos\left(\frac{n\pi}{2}\right)$

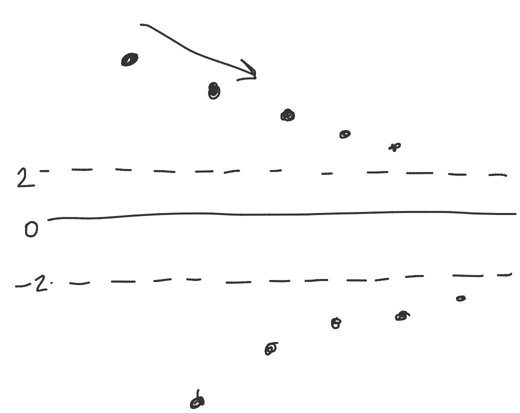
а) $x_1=5$, $x_2=-\left(2+\frac{3}{2}\right)$, $x_3=+\left(2+\frac{3}{3}\right)$, $x_4=-\left(2+\frac{3}{4}\right)$, ..., $x_5=+\left(2+\frac{3}{5}\right)$

$\quad\;\; \shortparallel$

$2+\frac{3}{1}$

$2+\frac{3}{n}>0$

$n\nearrow \Rightarrow \frac{3}{n}\searrow$



непарни су >0 и опадају, $\lim\limits_{n\to\infty}x_{2n-1}=2$

парни су <0 и расту (по $|\cdot|$ опадају), $\lim\limits_{n\to\infty}x_{2n}=-2$

$\sup\{x_n \mid n\in\mathbb{N}\}=x_1=5$

$\inf\{x_n \mid n\in\mathbb{N}\}=x_2=-\frac{7}{2}$

Т.Н. x_n су 2 и -2.

ПП. $\exists$ нека трећа Т.Н. α $(\alpha\neq 2,-2)$

Т.Н. x_n су $\underline{2 \text{ и } -2}$.

$$\underline{\lim_{n\to\infty}}\, x_n = -2$$

$$\overline{\lim_{n\to\infty}}\, x_n = 2$$

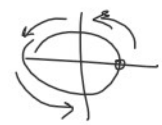
ПП. $\exists$ нека трећа Т.Н. α $(\alpha \neq 2, -2)$

$\Rightarrow \exists\, x_{n_k} \to \alpha$

x_{n_k} мора имати ∞ чланова у једном од подниза x_{2n} или x_{2n-1}

БУО нека је то x_{2n}

$\Rightarrow \quad \alpha = -2$ ↯



б) $x_n = 1 + \frac{n}{n+1}\cos\left(\frac{n\pi}{2}\right)$

$n = 4k$, $\cos\left(\frac{n\pi}{2}\right) = \cos(2k\pi) = 1$

$n = 4k+1$, $\cos = 0$

$n = 4k+2$, $\cos = -1$

$n = 4k+3$, $\cos = 0$

$$\boxed{\begin{aligned} x_{4k} &= 1 + \frac{4k}{4k+1} \\ x_{4k+1} &= \boxed{1} \\ x_{4k+2} &= 1 - \frac{4k+2}{4k+3} \\ x_{4k+3} &= \boxed{1} \end{aligned}}$$

Скуп вр. $A = \{x_n \mid n \in \mathbb{N}\} = \{1\} \cup \{\underbrace{1 + \frac{4k}{4k+1}}_{>1} \mid k \in \mathbb{N}\} \cup \{1 - \underbrace{\frac{4k+2}{4k+3}}_{0<\cdot<1} \mid k \in \mathbb{N}_0\}$

$$\frac{4k}{4k+1} = 1 - \frac{1}{4k+1}$$

$4k+1 \nearrow \Rightarrow \frac{1}{4k+1} \searrow \Rightarrow \frac{4k}{4k+1} \nearrow$

$\sup A = \lim\limits_{k\to\infty}\left(1 + \frac{4k}{4k+1}\right) = \boxed{2}$ (није max)

$\inf A = \lim\limits_{k\to\infty}\left(1 - \frac{4k+2}{4k+3}\right) = \boxed{0}$ (није min)

Т.Н. $\{0, \boxed{1}, 2\}$

$\boxed{\underline{\lim} = 0}$

$\boxed{\overline{\lim} = 2}$

→ оваква низ се зове Кошијев

конв. $\Leftrightarrow$ Коши.

$\boxed{\text{Т}}$ (x_n) конвергентан $\Leftrightarrow (\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall m, n \geq n_0)\ |x_m - x_n| < \varepsilon$.

$\left(\Leftrightarrow (\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)(\forall p \in \mathbb{N})\ |x_{n+p} - x_n| < \varepsilon. \quad (m = n+p)\right)$

lim ???

③ Доказати да је низ $x_n = \frac{\cos 1^2}{1\cdot 2} + \frac{\cos 2^2}{2\cdot 3} + \frac{\cos 3^2}{3\cdot 4} + \frac{\cos 4^2}{4\cdot 5} + \ldots + \frac{\cos n^2}{n\cdot(n+1)}$ конвергентан.

$x_n = \sum_{k=1}^{n} \frac{\cos k^2}{k(k+1)}$

$\dots \frac{\quad}{1\cdot 2}+\frac{\quad}{2\cdot 3}+\frac{\quad}{3\cdot 4}+\frac{\quad}{4\cdot 5}+\dots+\frac{\quad}{n\cdot(n+1)}$ конвергентан.

$$x_n=\sum_{k=1}^{n}\frac{\cos k^2}{k(k+1)}$$

$\varepsilon>0$ произв.

$$|x_{n+p}-x_n| = \left|\sum_{k=1}^{n+p}\frac{\cos k^2}{k(k+1)}-\sum_{k=1}^{n}\frac{\cos k^2}{k(k+1)}\right| = \left|\sum_{k=n+1}^{n+p}\frac{\cos k^2}{k(k+1)}\right| \le \sum_{k=n+1}^{n+p}\left|\frac{\cos k^2}{k(k+1)}\right| \le \sum_{k=n+1}^{n+p}\frac{1}{k(k+1)} =$$

$\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}$ $\qquad$ $|\cos k^2|\le 1$

$$=\left(\frac{1}{n+1}-\cancel{\frac{1}{n+2}}\right)+\left(\cancel{\frac{1}{n+2}}-\cancel{\frac{1}{n+3}}\right)+\left(\cancel{\frac{1}{n+3}}-\cancel{\frac{1}{n+4}}\right)+\dots+\left(\cancel{\frac{1}{n+p}}-\frac{1}{n+p+1}\right)=\frac{1}{n+1}\underbrace{-\frac{1}{n+p+1}}_{<0}<\frac{1}{n+1}<\varepsilon$$

$n\ge n_0$, где је $n_0=\left[\frac{1}{\varepsilon}\right]+1$. (Арх. св.)

lim??? – НЕ ЗНАМО!

4) $x_n=1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}$, $n\in\mathbb{N}$

– дивергира

Кошијев. крит.: $(\exists\varepsilon>0)(\forall n_0\in\mathbb{N})(\exists n\ge n_0)\boxed{(\exists p\in\mathbb{N})}\ |x_{n+p}-x_n|\ge\varepsilon$.

$p=n$: $$|x_{n+p}-x_n|=|x_{2n}-x_n|=\left|\left(\underbrace{1+\frac{1}{2}+\dots+\frac{1}{n}}_{x}+\frac{1}{n+1}+\dots+\frac{1}{2n}\right)-\left(\underbrace{1+\dots+\frac{1}{n}}_{x}\right)\right|$$

$$=\frac{1}{n+1}+\dots+\frac{1}{2n}\ge\underbrace{\frac{1}{2n}+\dots+\frac{1}{2n}}_{n}=n\cdot\frac{1}{2n}=\frac{1}{2}.\qquad \left(\varepsilon=\frac{1}{2}\right)$$

$\frac{1}{n+k}\ge\frac{1}{2n}$, $k\in\{1,\dots,n\}$

низ није Кошијев $\Rightarrow$ није конверг.

⊛ Доказати: а) $x_n=\frac{\sin 1!}{1^2}+\frac{\sin 2!}{2^2}+\frac{\sin 3!}{3^2}+\dots+\frac{\sin n!}{n^2}$ конв.

б) $x_n=\frac{1}{\log 2}+\frac{1}{\log 3}+\dots+\frac{1}{\log n}$ див.

Штолцова Т $\quad (b_n)\nearrow$, $b_n\to\infty$, a_n произв.

Ако $\exists\lim\limits_{n\to\infty}\frac{a_n-a_{n-1}}{b_n-b_{n-1}} \Rightarrow \exists\lim\limits_{n\to\infty}\frac{a_n}{b_n}$ и важи $\lim\limits_{n\to\infty}\frac{a_n}{b_n}=\lim\limits_{n\to\infty}\frac{a_n-a_{n-1}}{b_n-b_{n-1}}$

(доказ погл. у скрипти). (важи за низове у $\overline{\mathbb{R}}$)

(5) $L = \lim\limits_{n\to\infty} \dfrac{1+\sqrt{2}+\sqrt[3]{3}+\dots+\sqrt[n]{n}}{n} = ?$ $\sum \sqrt[k]{k} = ?$ ×

$$a_n = \underbrace{1+\sqrt{2}+\sqrt[3]{3}+\dots}_{a_{n-1}}+\sqrt[n]{n}$$

$$b_n = n$$

$$b_n \nearrow \infty \ \checkmark$$

$$\exists \lim \frac{a_n-a_{n-1}}{b_n-b_{n-1}} \ ?$$

$$\left.\begin{aligned} a_n-a_{n-1} &= \sqrt[n]{n} \\ b_n-b_{n-1} &= n-(n-1) = 1 \end{aligned}\right\} \lim_{n\to\infty} \frac{a_n-a_{n-1}}{b_n-b_{n-1}} = \lim \frac{\sqrt[n]{n}}{1} = 1.$$

$$\overset{\text{Штолц}}{\Longrightarrow} \exists \lim = L = 1.$$

(6) $\lim\limits_{n\to\infty} \dfrac{1^p+2^p+\dots+n^p}{n^{p+1}} = L$, $p\in\mathbb{N}_0$

$$a_n = 1^p+2^p+\dots+n^p$$

$$b_n = n^{p+1} \nearrow \infty$$

$$\frac{a_n-a_{n-1}}{b_n-b_{n-1}} = \frac{n^p}{n^{p+1}-\underbrace{(n-1)^{p+1}}_{\sum_{k=0}^{p+1}\binom{p+1}{k}\cdot n^k\cdot(-1)^{p+1-k}}} = \frac{n^p}{\cancel{n^{p+1}}-\left(\cancel{n^{p+1}}-\boxed{\binom{p+1}{1}n^p}+\binom{p+1}{2}n^{p-1}-\dots\right)}$$

није битно

$$\xrightarrow{n\to\infty} \frac{n^p}{\binom{p+1}{1}n^p} = \frac{1}{p+1}$$

Штолц $\Rightarrow$ $L = \dfrac{1}{p+1}$.

⊛ Домаћи: $\lim\limits_{n\to\infty}\left(\dfrac{1^p+2^p+\dots+n^p}{n^p} - \dfrac{n}{p+1}\right)$, $p\in\mathbb{N}$ (реш: $\frac{1}{2}$)

(7) Ако је a_n конвергентан и $\lim a_n = \alpha$, онда је $A_n = \dfrac{a_1+\dots+a_n}{n}$ такође конв. и $\lim A_n = \alpha$. [Кошијева теорема]

Код Штолца узмемо $x_n = a_1+\dots+a_n$, $y_n = n$ $(\nearrow\infty)$, $A_n = \dfrac{x_n}{y_n}$

$$\frac{x_n-x_{n-1}}{y_n-y_{n-1}} = \frac{a_n}{1} \to \alpha \overset{\text{Шт.}}{\Rightarrow} \lim A_n = \alpha.$$

(8) $(a_n)_{n\in\mathbb{N}}$ $a_n>0$, $\lim a_n = \alpha > 0$ $\Big\} \Rightarrow \lim \sqrt[n]{a_1 \cdot \ldots \cdot a_n} = \alpha.$

I начин: $H_n = \dfrac{n}{\frac{1}{a_1}+\ldots+\frac{1}{a_n}}$ – хармонијска средина

$G_n = \sqrt[n]{a_1\cdot\ldots\cdot a_n}$ – геометријска средина

$A_n = \dfrac{a_1+\ldots+a_n}{n}$ – аритметичка средина

ВАЖИ:
$$\boxed{H_n \le G_n \le A_n, \forall n}$$

$\lim G_n = ?$

(7) $\Rightarrow \lim A_n = \alpha.$

$$H_n = \frac{n}{\frac{1}{a_1}+\ldots+\frac{1}{a_n}} = \left(\frac{\frac{1}{a_1}+\ldots+\frac{1}{a_n}}{n}\right)^{-1} \longrightarrow \left(\frac{1}{\alpha}\right)^{-1} = \alpha$$

$$\left(a_n \to \alpha \underset{(a_n>0)}{\Rightarrow} \frac{1}{a_n} \to \frac{1}{\alpha} \overset{(7)}{\Rightarrow} \frac{\frac{1}{a_1}+\ldots+\frac{1}{a_n}}{n} \to \frac{1}{\alpha}\right)$$

Тодп: $\left.\begin{matrix} A_n \to \alpha \\ H_n \to \alpha \end{matrix}\right\} G_n \to \alpha.$

II начин: $b_n = \log(a_n)$

$a_n \to \alpha \Rightarrow b_n \to \log(\alpha)$

$\big\lceil \log a^b = b\cdot \log a$
$\log(a\cdot b) = \log a + \log b \big\rfloor$

$$\lim(\log(G_n)) = \lim\left(\log(a_1\cdot\ldots\cdot a_n)^{\frac{1}{n}}\right) = \lim\left(\frac{1}{n}\log(a_1\cdot\ldots\cdot a_n)\right) =$$
$$= \lim \frac{\log(a_1)+\ldots+\log(a_n)}{n} =$$
$$= \lim\frac{b_1+\ldots+b_n}{n} \overset{(7)}{=} \log(\alpha).$$

$\parallel$

$\log(\lim G_n)$

$\Rightarrow \lim G_n = \alpha.$

(9) $(x_n)_{n\in\mathbb{N}}$, $x_n>0$ и $\boxed{\exists \lim \frac{x_n}{x_{n-1}}}$

Тогда $\exists \lim \sqrt[n]{x_n}$ и $\lim \sqrt[n]{x_n} = \lim \dfrac{x_n}{x_{n-1}}$.

$$\left.\begin{array}{l} \boxed{y_n = \dfrac{x_n}{x_{n-1}}} \\ y_1 = x_1 \end{array}\right\} \quad x_1, \frac{x_2}{x_1}, \frac{x_3}{x_2}, \ldots, \frac{x_n}{x_{n-1}}$$

теор. ⑧

$$\boxed{\exists \lim y_n} \Rightarrow \exists \lim \sqrt[n]{y_1 \cdot \ldots \cdot y_n} = \lim y_n$$

$$\sqrt[n]{y_1 \ldots y_n} = \sqrt[n]{\cancel{x_1} \cdot \frac{\cancel{x_2}}{\cancel{x_1}} \cdot \frac{\cancel{x_3}}{\cancel{x_2}} \cdot \ldots \cdot \frac{x_n}{\cancel{x_{n-1}}}} = \sqrt[n]{x_n}.$$

⑩ Доказать $\lim\limits_{n\to\infty} \dfrac{n}{\sqrt[n]{n!}} = e$.

$$\frac{n}{\sqrt[n]{n!}} = \sqrt[n]{\frac{n^n}{n!}} \quad \frac{n^n}{n!} = x_n \quad = \sqrt[n]{x_n}$$

$\lim\limits_{n\to\infty} \sqrt[n]{x_n} = ?$

$$?\exists \lim \frac{x_n}{x_{n-1}} = \lim \frac{\frac{n^n}{n!}}{\frac{(n-1)^{n-1}}{(n-1)!}} = \lim \frac{\cancel{(n-1)!}}{n\cancel{!}} \cdot \frac{n^n}{(n-1)^{n-1}} =$$

$$= \lim \frac{n^{n-1}}{(n-1)^{n-1}} = \lim \left(\frac{n}{n-1}\right)^{n-1} = \lim \left(1 + \frac{1}{n-1}\right)^{n-1} = e$$

⑨ $\Rightarrow \lim \sqrt[n]{x_n} = \lim \dfrac{n}{\sqrt[n]{n!}} = e$.