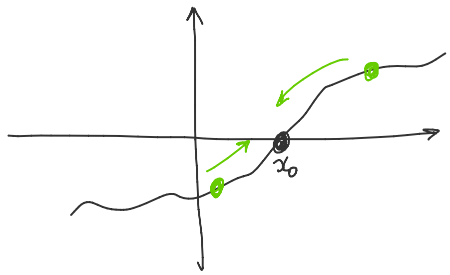
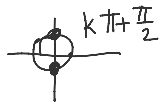
Понедељак 21.12.2020.

① $f(x)=5x\cdot\log(\operatorname{tg}x)+x-5$. Доказати да f има реалну нулу.

$f(x_0)=0$.

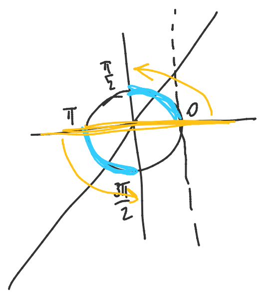


$\operatorname{tg}x=\frac{\sin x}{\cos x}$



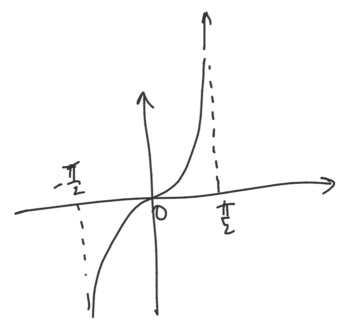
$x\neq k\pi+\frac{\pi}{2},\ k\in\mathbb{Z}$

$\operatorname{tg}x>0$



$x\in\left(k\pi,k\pi+\frac{\pi}{2}\right),\ k\in\mathbb{Z}$

$Dom f=\bigcup_{k\in\mathbb{Z}}\left(k\pi,k\pi+\frac{\pi}{2}\right)$



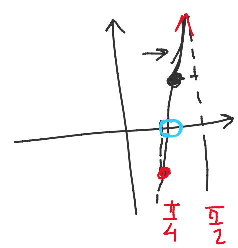
Посматрамо f на $\left(0,\frac{\pi}{2}\right)$: $\lim_{x\to 0+} f(x) =$ "$0\cdot\log(0+)-5$" ?

са десне стране

$$\lim_{x\to\frac{\pi}{2}-} f(x)=\lim_{x\to\frac{\pi}{2}-}\Big(\underbrace{5\cdot\underbrace{x}_{\to\frac{\pi}{2}}\cdot\underbrace{\log(\underbrace{\operatorname{tg}x}_{\to+\infty})}_{\to+\infty}}_{+\infty}+\underbrace{x}_{\to\frac{\pi}{2}}-5\Big)=+\infty$$

$\operatorname{tg}\frac{\pi}{4}=1$

$$f\left(\frac{\pi}{4}\right)=5\cdot\frac{\pi}{4}\cdot\underbrace{\log(1)}_{0}+\frac{\pi}{4}-5=\frac{\pi}{4}-5<0$$



$$\lim_{x\to a} f(x)=+\infty \iff (\forall M>0)(\exists\delta>0)(\forall x\in\mathbb{R})\ |x-a|<\delta\Rightarrow f(x)>M$$

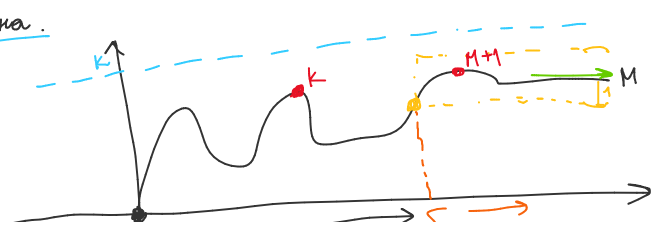
околина $+\infty$

$$\Rightarrow\left(\exists c\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)\right)\ f(c)=0.$$

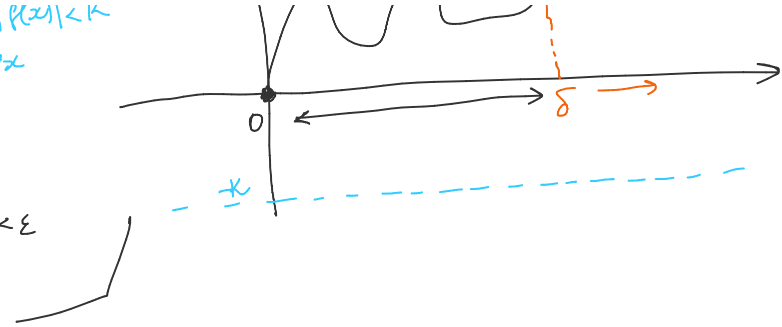
② f непрекидна и $\lim_{x\to+\infty} f(x)=M\in\mathbb{R}$. Доказати да је f ограничена.

$f:[0,+\infty)\to\mathbb{R}$

$(\exists K>0)\ |f(x)|<K\ \forall x$



$(\exists K>0)\ |f(x)|<K$
$\forall x$



T $\lim\limits_{x\to+\infty} f(x)=M \iff (\forall\varepsilon>0)(\exists\delta>0)(\forall x\in\mathbb{R})\ x>\delta \Rightarrow |f(x)-M|<\varepsilon$

$\varepsilon=1:\ (\exists\delta>0)\quad x>\delta \Rightarrow |f(x)-M|<1$

$\downarrow$

$f(x)\in(M-1,M+1) \longrightarrow |f(x)|\le \max\{|M-1|,|M+1|\}$

на $[0,\delta]$? $\longrightarrow$ Вајерштрасова Т

$(\exists c_1,c_2\in[0,\delta])\quad f(c_1)=\min\limits_{[0,\delta]} f$, f ограничена на $[0,\delta]$

$f(c_2)=\max\limits_{[0,\delta]} f$ $\qquad |f(x)|\le K,\ x\in[0,\delta]$

На $[0,+\infty)$: $\quad |f(x)|\le \max\{\underset{x\in[0,\delta]}{K}, \underbrace{|M-1|, |M+1|}_{x>\delta}\}$

T кратко: повратак на inf и sup

(3) $A=\left\{\frac{m}{|m|+n}\ \middle|\ m\in\mathbb{Z}, n\in\mathbb{N}\right\}\subseteq\mathbb{R}$, $\inf A, \sup A$?

$|m|\ge 0$
$|m|+n\ge 1$

Докажимо $\sup A=1$.

1) $\frac{m}{|m|+n}\le 1 \iff m\le|m|+n$ ✓ (јер $|m|\ge m$, $n\ge 1>0$)

2) $(\forall\varepsilon>0)$ $1-\varepsilon$ није ГО од A

$\iff$

$(\forall\varepsilon>0)(\exists a\in A)\quad a>1-\varepsilon$

$\varepsilon > 0$ произв.

$$\frac{m}{|m|+n} > 1-\varepsilon \quad , \quad \left.\begin{matrix} m\in\mathbb{Z} \\ n\in\mathbb{N}\end{matrix}\right\} \text{ да ли } \exists ?$$

$n=1$
$m>0$

$$\frac{m}{m+1} > 1-\varepsilon \iff 1-\frac{1}{m+1} > 1-\varepsilon \iff \frac{1}{m+1} < \varepsilon \checkmark$$

$$\left[\frac{m}{m+1} = \frac{m+1-1}{m+1} = 1-\frac{1}{m+1}\right]$$

по Архимедовој аксиоми

$$\underbrace{(m+1)}_{\in\mathbb{N}} \cdot \underbrace{\varepsilon}_{a} > \underbrace{1}_{b}$$

Доказујемо $\inf A = -1$:

1) $-1 \leq \frac{m}{|m|+n} \quad / \cdot (|m|+n) \qquad (-1 \text{ је ДО})$

$$\iff -|m|-n \leq m \iff |m|+n \geq -m \checkmark \quad \left(\begin{matrix}|m| \geq -m \\ n \geq 1 \geq 0\end{matrix}\right)$$

2) $(\forall \varepsilon > 0) \quad -1+\varepsilon$ није ДО

$$\iff$$

$$(\forall\varepsilon>0)(\exists m\in\mathbb{Z})(\exists n\in\mathbb{N}) \quad \frac{m}{|m|+n} < -1+\varepsilon$$

$$\left.\begin{matrix} m<0 \; / |m| = -m) \\ n=1 \end{matrix}\right\} \quad \frac{m}{1-m} < -1+\varepsilon$$

$$\frac{m+(1-m)}{1-m} < \varepsilon$$

$$\frac{1}{1-m} < \varepsilon \Rightarrow \varepsilon \cdot (1-m) > 1 \quad (\text{Арх} \checkmark)$$

$1-m > 0$

④ а) $\inf\limits_{n\in\mathbb{N}} \sup\limits_{m\in\mathbb{N}} \frac{m-n}{m+n}$

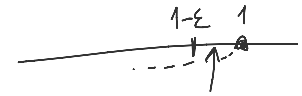
б) $\sup\limits_{m\in\mathbb{N}} \inf\limits_{n\in\mathbb{N}} \frac{m-n}{m+n}$

в)
г) скриптa

a) $\sup\limits_{m\in\mathbb{N}} \frac{m-n}{m+n} = f(n) \leftarrow \inf\limits_{n\in\mathbb{N}}$

→ фиксирамо $n\in\mathbb{N}$.

$m+n \geq 1+1 = 2$



$$\sup_{m\in\mathbb{N}} \frac{m-n}{m+n} = 1$$

1) $\frac{m-n}{m+n} \leq 1 \iff m-n \leq m+n \iff 0 \leq 2n$ ✓

2) $(\forall \varepsilon>0) ? (\exists m\in\mathbb{N})\ \frac{m-n}{m+n} > 1-\varepsilon \ / \cdot (m+n)$

$\cancel{m}-n > \cancel{m}+n-\varepsilon(m+n)$

$\varepsilon(m+n) > 2n$

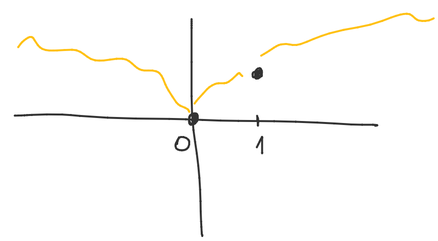
$\underbrace{m}_{\in\mathbb{N}} \cdot \underbrace{\varepsilon}_{a} > \underbrace{(2n-\varepsilon n)}_{b}$ (Арх)

✓

$$\inf_{n\in\mathbb{N}} \sup_{m\in\mathbb{N}} \frac{m-n}{m+n} = \inf_{n\in\mathbb{N}} 1 = 1$$

⑤ Испитати непрекидност фје

$$f(x) = \begin{cases} \frac{1}{1-e^{\frac{x}{1-x}}}, & x \notin \{0,1\} \\ 0, & x=0 \\ 1, & x=1 \end{cases}$$



f је непр у $x=a$ акко $\lim\limits_{x\to a} f(x) = f(a)$.

$\frac{x}{1-x}$ није деф у $x=1$

$x=0$: $1-e^{\frac{x}{1-x}} = 1-e^0 = 0$ → дељење са 0

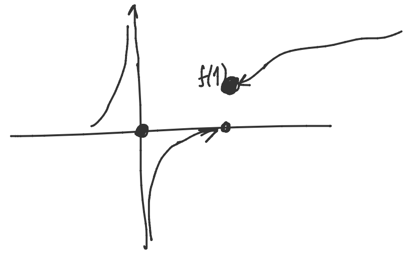
1° $x \notin \{0,1\}$

$f(x) = \frac{1}{1-e^{\frac{x}{1-x}}}$ → f је непрекидна као композиција непр. фја

2° $x=0$

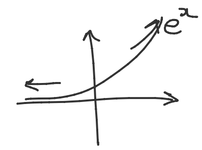
$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{1}{1-e^{\frac{x}{1-x}}} = -\infty$$

$\frac{x}{1-x} \to 0+ \Rightarrow e^{\frac{x}{1-x}} \to 1+ \Rightarrow 1-e^{\frac{x}{1-x}} \to 0- \Rightarrow \frac{1}{---} \to -\infty$



$$\lim_{x\to 0-} f(x) = +\infty$$

$\Rightarrow f$ има прекид у $x=0$



3° $x=1$

$$\lim_{x\to 1+} f(x) = 1$$

$$\frac{x}{1-x} \to -\infty \Rightarrow e^{\frac{x}{1-x}} \to 0 \Rightarrow \frac{1}{1-e^{\frac{x}{1-x}}} \to 1$$

$$\lim_{x\to 1-} f(x) = 0$$

$$\frac{x}{1-x} \to +\infty \Rightarrow e^{\frac{x}{1-x}} \to +\infty \Rightarrow \frac{1}{\underbrace{1-e^{\frac{x}{1-x}}}_{\to -\infty}} \to 0$$

$$\nexists \lim_{x\to 1} f(x) \quad \Rightarrow f \text{ има прекид у } x=1$$

⑥ Испитати непр.

$$g(x) = \begin{cases} 0, & x=0 \\ \sqrt{|x|}\cdot \sin\frac{1}{x}, & x\neq 0 \end{cases}$$

1° $x\neq 0$: композ. непр $\Rightarrow g$ непр.

2° $x=0$: $\lim\limits_{x\to 0} \sqrt{|x|}\cdot\sin\frac{1}{x} = ?$

$-1 \le \sin(\cdot) \le 1$

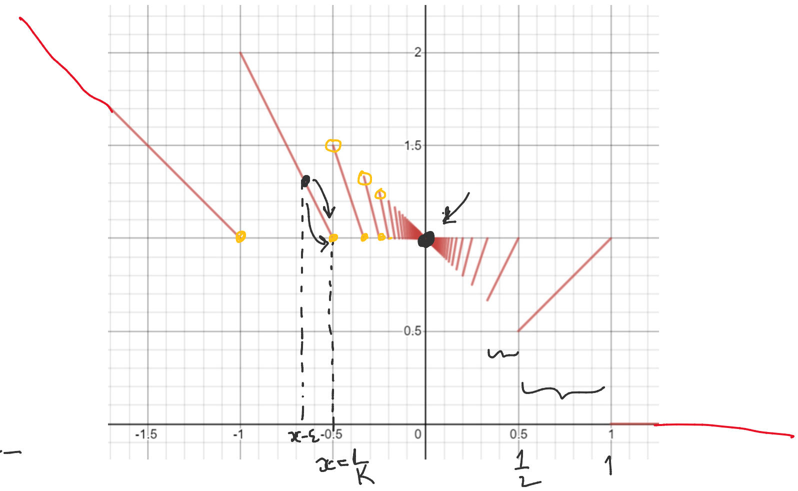
$$-\sqrt{|x|} \le \underbrace{\sqrt{|x|}}_{\ge 0}\cdot \sin\frac{1}{x} \le \sqrt{|x|}$$

$$\left.\begin{aligned} &\lim_{x\to 0}\left(-\sqrt{|x|}\right) = -\sqrt{|0|} = 0 \\ &\lim_{x\to 0}\sqrt{|x|} = \sqrt{|0|} = 0 \end{aligned}\right\} \Rightarrow$$

Т о 3 милицајца

$$\lim_{x\to 0} \underbrace{\sqrt{|x|}\cdot\sin\frac{1}{x}}_{g(x)} = 0 = g(0) \Rightarrow g \text{ је непр. у } x=0$$

(7) $h(x)=\begin{cases} 1, & x=0 \\ x\cdot\left[\frac{1}{x}\right], & x\neq 0 \end{cases}$



$\left[\frac{1}{x}\right]$

$\frac{1}{x}\in\mathbb{Z} \Rightarrow x=\frac{1}{k}, k\in\mathbb{Z}$

1° $x\in\left(\frac{1}{k+1},\frac{1}{k}\right)$, $k>0$, $k<-1 \rightarrow \left(-1,-\frac{1}{2}\right),\left(-\frac{1}{2},-\frac{1}{3}\right),\ldots$

$\left(\frac{1}{2},1\right),\left(\frac{1}{3},\frac{1}{2}\right),\ldots$

$h(x)=c\cdot x$ → непр.

2° $x=0$

$\frac{1}{x}-1\leq\left[\frac{1}{x}\right]\leq\frac{1}{x} \Rightarrow$

1) $x>0$ $\quad 1-x\leq x\cdot\left[\frac{1}{x}\right]\leq 1$

$x\to 0+$: $\lim\limits_{x\to 0+}(1-x)=\lim\limits_{x\to 0+}1=1 \overset{\text{ТОЗЛ}}{\Longrightarrow} \lim\limits_{x\to 0+}x\cdot\left[\frac{1}{x}\right]=1$

2) $x<0$

$\frac{1}{x}-1\leq\left[\frac{1}{x}\right] \;/\cdot x$ $\qquad \left[\frac{1}{x}\right]\leq\frac{1}{x}\;/\cdot x$

$1-x\geq x\cdot\left[\frac{1}{x}\right]$ $\qquad x\cdot\left[\frac{1}{x}\right]\geq 1$

$1-x\geq x\cdot\left[\frac{1}{x}\right]\geq 1$

$\lim\limits_{x\to 0-}(1-x)=\lim\limits_{x\to 0-}1=1 \Rightarrow \lim\limits_{x\to 0-}x\cdot\left[\frac{1}{x}\right]=1$

$\lim\limits_{x\to 0}\underbrace{x\cdot\left[\frac{1}{x}\right]}_{h(x)}=1=h(0)$

$\Downarrow$

h непр у $x=0$

3° $x=\frac{1}{k}$, $k\in\mathbb{Z}\setminus\{0\}$ (нпр радимо $k>0$, исто је $k<0$)

→ имамо прекид

$\lim\limits_{x\to\frac{1}{k}-}x\cdot\left[\frac{1}{x}\right]=\lim\limits_{\varepsilon\to 0+}h\left(\frac{1}{k}-\varepsilon\right)=\lim\limits_{\varepsilon\to 0+}\left(\frac{1}{k}-\varepsilon\right)\cdot\left[\frac{1}{\frac{1}{k}-\varepsilon}\right]\underset{\lim\limits_{\varepsilon\to 0+}}{=}\left(\frac{1}{k}-\varepsilon\right)\cdot\left[\frac{k}{1-\varepsilon\cdot k}\right]\underset{\lim\limits_{\varepsilon\to 0+}}{=}\left(\frac{1}{k}-\varepsilon\right)\cdot k=\lim\limits_{\varepsilon\to 0+}(1-\varepsilon k)=1$

$\left(\frac{1}{k+1},\frac{1}{k}\right)$ $\qquad \frac{1}{k}-\varepsilon\in\left(\frac{1}{k+1},\frac{1}{k}\right)$

$\varepsilon<\frac{1}{k}-\frac{1}{k+1}=\frac{k+1-k}{(k+1)k}=\frac{1}{k(k+1)} \Rightarrow 1-\varepsilon\cdot k>1-\frac{1}{k+1}=\frac{k}{k+1}$

$k \quad k \quad k+1 \Rightarrow \left[\frac{k}{\ldots}\right]=k$

$$\varepsilon < \frac{1}{k} - \frac{1}{k+1} = \frac{k+1-k}{(k+1)k} = \frac{1}{k(k+1)} \Rightarrow 1-\varepsilon\cdot k > 1 - \frac{\ }{k+1} = \frac{\ }{k+1}$$

$$k < \frac{k}{\underbrace{1-\varepsilon\cdot k}_{>0}} < \frac{k}{\frac{k}{k+1}} = k+1 \Rightarrow \left[\frac{k}{1-\varepsilon k}\right] = k$$

$$\lim_{x\to \frac{1}{k}+} h(x) = \lim_{\varepsilon\to 0+} h\left(\frac{1}{k}+\varepsilon\right) = \lim_{\varepsilon\to 0+} \left(\frac{1}{k}+\varepsilon\right)\cdot\left[\frac{1}{\frac{1}{k}+\varepsilon}\right] = \lim_{\varepsilon\to 0+}\left(\frac{1}{k}+\varepsilon\right)\cdot\left[\frac{k}{1+\varepsilon\cdot k}\right] = \lim_{\varepsilon\to 0+} \boxed{\frac{1}{k}(k-1)} + \underbrace{\varepsilon\cdot(k-1)}_{\to 0} = \frac{k-1}{k} \neq 1$$

$$\frac{1}{k}+\varepsilon \in \left(\frac{1}{k}, \frac{1}{k-1}\right)$$

$$\varepsilon < \frac{1}{k-1} - \frac{1}{k} = \frac{1}{k(k-1)}$$

$$1 < 1+\varepsilon\cdot k < 1 + \frac{1}{k-1}$$

$$k > \frac{k}{1+\varepsilon\cdot k} > \frac{k}{1+\frac{1}{k-1}} = k-1 \Rightarrow \left[\frac{k}{1+\varepsilon k}\right] = k-1$$

$$\left.\begin{array}{l} \lim\limits_{x\to \frac{1}{k}-} h(x) = 1 \\ \qquad\qquad \neq \\ \lim\limits_{x\to \frac{1}{k}+} h(x) = \frac{k-1}{k} \end{array}\right\} \Rightarrow \nexists \lim_{x\to\frac{1}{k}} h(x) \Rightarrow h \text{ није непр. у } x = \frac{1}{k}$$