Петак 21.05.2021.

1) Условну и апсолутну конвергенцију $\int\limits_0^{+\infty} x^2\cos(e^x)\,dx$

$+\infty$ једини синг.

$$\int\limits_0^{+\infty} x^2\cos(e^x)\,dx \ (*) = \left/ \begin{array}{l} e^x = t \\ x=\log t \\ dx = \frac{dt}{t} \end{array} \quad \begin{array}{c|c|c} x & 0 & +\infty \\ t & 1 & +\infty \end{array} \right/ = \int\limits_1^{+\infty} (\log t)^2\cdot\cos(t)\cdot\frac{dt}{t} = \int\limits_1^{+\infty}\frac{\log^2 t}{t}\cdot\cos t\,dt$$

Хоћемо Дирихлеа:

Д1) $\left|\int\limits_0^x \cos t\,dt\right| = \left|\sin t\Big|_0^x\right| = |\sin x| \le 1$

Д2) $g(t) = \frac{\log^2 t}{t}$

$$\lim_{t\to\infty} g(t) = \lim_{t\to\infty}\frac{\log^2 t}{t} = 0.$$

(2 пута Лопитал)

g монотона?

$$g'(t) = \frac{2\log t\cdot\frac{1}{t}\cdot t - \log^2 t\cdot 1}{t^2} = \frac{2\log t - \log^2 t}{t^2} = \frac{\log t}{t^2}\cdot(2-\log t) < 0, \quad t > e^2$$

$\Rightarrow g(t)\downarrow$ за $t > e^2$

$\xRightarrow{\text{Дир.}} \int\limits_{e^2}^{+\infty}\frac{\log^2 t}{t}\cos t\,dt$ конв.

$\int\limits_1^{e^2}$ је својствен тј. интеграл је Риманов

$\int\limits_1^{+\infty}\frac{\log^2 t}{t}\cos t\,dt$ конв. $\Rightarrow$ почетни $(*)$ конв.

Апсолутна конв?
$$\int\limits_0^{\infty}|x^2\cos(e^x)|\,dx \ (*) = \dots = \int\limits_1^{+\infty}\left|\frac{\log^2 t}{t}\cos t\right|dt = \int\limits_1^{+\infty}\frac{\log^2 t}{t}\cdot|\cos t|\,dt \geq \int\limits_e^{+\infty}\frac{\log^2 t}{t}\cdot|\cos t|\,dt$$

$t \geq e$: $\quad \frac{\overset{\geq 1}{\log^2 t}}{t}\cdot|\cos t| \geq \frac{|\cos t|}{t} \geq \frac{\cos^2 t}{t} = \frac{1+\cos 2t}{t} = \frac{1}{t} + \frac{\cos 2t}{t}$

$\left.\begin{array}{l} \int\limits_e^{+\infty}\frac{1}{t}\,dt \leftarrow \text{див} \\ \int\limits_e^{+\infty}\frac{\cos 2t}{t} \leftarrow \text{конв. (Дирихле)} \end{array}\right\} \Rightarrow \int\limits_e^{+\infty}\frac{\cos^2 t}{t}\,dt$ див. $\xRightarrow{\text{ПК}} \int\limits_e^{+\infty}\frac{\log^2 t}{t}|\cos t|\,dt$ див.

$\xRightarrow{\text{ПК}}$ почетни $(*)$ див.

$\int_e^{+\infty} \frac{\cos 2t}{t} \leftarrow$ конв. (Дирихле)

$\overset{ПК}{\Longrightarrow}$ почетни $\overset{(*)}{}$ див.

$\Rightarrow$ почетни $\overset{(*)}{}$ условно конв.

② У зависности од $\alpha \in \mathbb{R}$ испитати конв. $\int_0^{+\infty} \cos(x^\alpha)\,dx$?

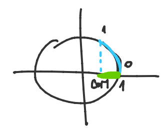
1° $\alpha = 0$ $\quad \int_0^{+\infty} \cos 1\,dx = \infty$ див.

2° $\alpha < 0$ $\quad$ синг. 0 и $+\infty$

$\downarrow$

$\frac{1}{x^{-\alpha}}$

$$\int_0^{+\infty} = \int_0^1 + \int_1^{+\infty}$$



$\int_1^{+\infty}$: $x \geq 1 \Rightarrow x^{-\alpha} \geq 1 \Rightarrow 0 \leq x^\alpha \leq 1 \Rightarrow \cos(x^\alpha) \geq 0$

$(-\alpha > 0)$ $\qquad \in [\cos 1, 1]$

$$\cos(x^\alpha) \sim \cos 0 = 1 \quad , x \to \infty$$

$$\int_1^{+\infty} 1\,dx \text{ див} \overset{ПК}{\Longrightarrow} \int_1^{+\infty} \cos(x^\alpha)\,dx \text{ див.}$$

$\int_0^1 \cos(x^\alpha)\,dx$? $\qquad |\cos(x^\alpha)| \leq 1$

$\int_0^1 1\,dx = 1$ конв. $\overset{ПК}{\Longrightarrow} \int_0^1 |\cos(x^\alpha)|\,dx$ конв. $\Rightarrow \int_0^1 \cos(x^\alpha)\,dx$ апс. конв. $\Rightarrow$ обично конв.

$$\Rightarrow \int_0^{+\infty} \cos(x^\alpha)\,dx \text{ див.}$$

3° $\alpha > 0$

$$\int_0^{+\infty} \cos(x^\alpha)\,dx = \left| \begin{array}{l} x^\alpha = t \\ x = t^{1/\alpha} \end{array} \quad dx = \frac{1}{\alpha} \cdot t^{1/\alpha - 1}\,dt \quad \begin{array}{c|c|c} x & 0 & +\infty \\ \hline t & 0 & +\infty \end{array} \right|$$

$$= \int_0^{+\infty} \cos t \cdot \frac{1}{\alpha} \cdot t^{1/\alpha - 1}\,dt = \frac{1}{\alpha} \cdot \int_0^{+\infty} \frac{\cos t}{t^{1-\frac{1}{\alpha}}}\,dt$$

3.1° $1 - \frac{1}{\alpha} > 0 \Rightarrow \frac{1}{\alpha} < 1 \Rightarrow \underline{\underline{\alpha > 1}}$: $\int_0^{+\infty} \frac{\cos t}{t^{1-\frac{1}{\alpha}}}\,dt$ (условно) конв. по Дирихлеу

3.2° $\alpha=1$ $\quad \displaystyle\int_0^{+\infty} \frac{\cos t}{t^{1-1}}\,dt = \int_0^{+\infty} \cos t\,dt = \lim_{M\to\infty} \sin M \nexists$ див.

3.3° $\underline{\alpha\in(0,1)}$

Кошијев критеријум конвергенције за несвојствене интеграле:

b-синг.

$$\int_a^b f(x)\,dx \text{ конв.} \iff (\forall \varepsilon>0)(\exists b_0\in(a,b))(\forall c,d\in(b_0,b)) \quad \left|\int_c^d f(x)\,dx\right| < \varepsilon$$

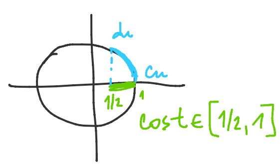
$$\int_a^b f(x)\,dx \text{ див.} \iff (\exists \varepsilon>0)(\forall b_0\in(a,b))(\exists c,d\in(b_0,b)) \quad \left|\int_c^d f(x)\,dx\right| \geq \varepsilon$$

$\displaystyle\int_0^{+\infty} \cos t \cdot t^{\frac{1}{\alpha}-1}\,dt$, хоћемо да $\underline{\underline{\text{див.}}}$ по Кошију

$\left.\begin{array}{l}\forall b_0\in(0,+\infty)\\ \exists c,d\geq b_0\end{array}\right\}$ узмемо низове c_n, d_n тд. $c_n, d_n \to \infty$.

$$c_n = 2n\pi$$
$$d_n = 2n\pi + \frac{\pi}{3}$$

$$\left|\int_{c_n}^{d_n} \underbrace{\cos t \cdot t^{\frac{1}{\alpha}-1}}_{\substack{\geq 0 \\ \text{на } [c_n,d_n]}}\,dt\right| = \int_{2n\pi}^{2n\pi+\frac{\pi}{3}} \underbrace{\cos t}_{\geq \frac{1}{2}} \cdot t^{\frac{1}{\alpha}-1}\,dt \geq \frac{1}{2}\cdot\int_{2n\pi}^{2n\pi+\frac{\pi}{3}} t^{\frac{1}{\alpha}-1}\,dt = \frac{1}{2}\cdot \left.\frac{t^{\frac{1}{\alpha}}}{\frac{1}{\alpha}}\right|_{2n\pi}^{2n\pi+\frac{\pi}{3}} =$$



$$(1+x)^\alpha - 1 \sim \alpha x$$

$$= \frac{1}{2\alpha}\cdot\left(\left(2n\pi+\frac{\pi}{3}\right)^{\frac{1}{\alpha}} - (2n\pi)^{\frac{1}{\alpha}}\right) = \frac{(2n\pi)^{\frac{1}{\alpha}}}{2\alpha}\cdot\left(\left(1+\frac{1}{6n}\right)^{\frac{1}{\alpha}} - 1\right) \sim$$

$$\sim \frac{(2n\pi)^{\frac{1}{\alpha}}}{2\alpha}\cdot\frac{1}{6n\alpha} = \frac{(2\pi)^{\frac{1}{\alpha}}}{12\alpha^2}\cdot n^{\overbrace{\frac{1}{\alpha}-1}^{>0}} \xrightarrow[n\to\infty]{} \infty$$

тј. $\left|\displaystyle\int_{c_n}^{d_n} \dots\right| \geq 1$, почевши од неког n

$\varepsilon=1$ узмемо

$\Rightarrow$ див. по Кошију

Закључак: конв. $\Leftrightarrow$ $\alpha > 1$.

Разни задаци (погледати и код Данила)

① Доказати да је $\displaystyle\int_0^{+\infty} \frac{\log x \, dx}{\sqrt{x}(1+x)} = 0$.

$$\int_0^{+\infty} = \int_0^1 + \int_1^{+\infty}$$

$$\int_1^{+\infty} \frac{\log x \, dx}{\sqrt{x}(1+x)} = \left| \begin{array}{l} x = \frac{1}{t} \\ dx = -\frac{dt}{t^2} \end{array} \quad \begin{array}{c|c|c} x & 1 & +\infty \\ t & 1 & 0 \end{array} \right| = \int_1^0 \frac{\overbrace{\log\left(\frac{1}{t}\right)}^{-\log t} \cdot \left(-\frac{dt}{t^2}\right)}{\frac{1}{\sqrt{t}} \cdot \left(1 + \frac{1}{t}\right)} = \int_1^0 \frac{\log t \cdot \frac{dt}{t^2}}{\frac{\sqrt{t}}{t} \cdot \frac{t+1}{t}} = -\int_0^1 \frac{\log t \, dt}{\sqrt{t}(1+t)}$$

$$\Rightarrow \int_0^{+\infty} = \int_0^1 + \int_1^{+\infty} = \int_0^1 - \int_0^1 = 0$$

морају оба да конв!

$$\int_1^{+\infty} \frac{\log x \, dx}{\sqrt{x}(1+x)} \qquad \frac{\log x}{\sqrt{x}(1+x)} \sim \frac{\log x}{\sqrt{x} \cdot x} = \frac{1}{x^{3/2} \cdot (\log x)^{-1}}, \; x \to +\infty$$

$$\overset{\text{ПК}}{\Longrightarrow} \left(\tfrac{3}{2} > 1\right) \text{ конв.}$$

и други конв. због смене $\Rightarrow$ заиста је 0

② Испитати усл. и апс. конв. реда $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{\sqrt{n}} \cdot \underbrace{\int_0^n e^{-x^2} dx}_{\text{зависи од } n}$.

$$a_n = (-1)^n \cdot \frac{1}{\sqrt{n}}$$

$$b_n = \int_0^n e^{-x^2} dx$$

Абел!

A1) $\sum a_n$ конв. $\quad \sum (-1)^n \cdot \frac{1}{\sqrt{n}}$, $\frac{1}{\sqrt{n}} \searrow 0 \Rightarrow$ (Лајбниц) конв.

А2) b_n монотон и огр.

$$b_{n+1}-b_n=\int_0^{n+1}e^{-x^2}dx-\int_0^{n}e^{-x^2}dx=\int_n^{n+1}\underbrace{e^{-x^2}}_{\geq 0}dx\geq 0 \Rightarrow b_{n+1}\geq b_n \Rightarrow b_n\nearrow$$

$$0\leq b_n=\int_0^n e^{-x^2}dx \underset{n\to\infty}{\leq} \int_0^{+\infty}e^{-x^2}dx$$

анализа 2: $\int_0^{+\infty}e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$

$$\int_0^{+\infty}e^{-x^2}dx=\underbrace{\int_0^1 \underbrace{e^{-x^2}}_{\leq 1}dx}_{\leq 1}+\int_1^{+\infty}e^{-x^2}dx\leq 1+\int_1^{+\infty}e^{-x}dx=1+\left(-e^{-x}\right)\Big|_1^{+\infty}=1+e^{-1}=1+\frac{1}{e}$$

$e^{-x^2}\leq e^{-x}$

$\Leftrightarrow$

$-x^2\leq -x$

$x^2\geq x$

$x\geq 1$

знамо да израч.

$\Rightarrow b_n$ огр. $\Rightarrow$ ред конв.

Апсолутна конв.?

$$\sum_{n=1}^{\infty}\left|(-1)^n\underbrace{\frac{1}{\sqrt{n}}}_{\geq 0}\underbrace{\int_0^n e^{-x^2}dx}_{\geq 0}\right|=\sum\frac{1}{\sqrt{n}}\cdot\int_0^n e^{-x^2}dx$$

идеја: $\int_0^n e^{-x^2}dx$ превише расте!

$$\int_0^n e^{-x^2}dx\longrightarrow\int_0^{+\infty}e^{-x^2}dx=M\ \left(=\frac{\sqrt{\pi}}{2}\right)$$

$$\frac{1}{\sqrt{n}}\cdot\int_0^n e^{-x^2}dx\sim\frac{1}{\sqrt{n}}\cdot M=\frac{M}{n^{1/2}}$$

$\sum\frac{M}{n^{1/2}}$ див. $\left(\frac{1}{2}<1\right)$ $\overset{ПК}{\Rightarrow}$ почетни само условно конв.

③ Низ $(x_n)_{n\in\mathbb{N}}$ задат $x_0>0$, $x_{n+1}=\dfrac{x_n}{x_n^2+x_n+1}$. Нађи $\lim\limits_{n\to\infty}(n\cdot x_n)$.

Индукцијом $x_n>0$. (огр. одоздо)

$$x_{n+1}=\frac{x_n}{\underbrace{x_n^2+x_n+1}_{\geq 1}}\leq\frac{x_n}{1}=x_n\Rightarrow x_n\searrow$$

монотон. опад.

$\Rightarrow \exists\lim x_n=\alpha$

$$x_{n+1} = \frac{x_n}{1+x_n+x_n^2} \quad \Big/ \lim_{n\to\infty}$$

$$\alpha = \frac{\alpha}{1+\alpha+\alpha^2} \Rightarrow \alpha(1+\alpha+\alpha^2) - \alpha = 0$$

$$\alpha^2(\alpha+1) = 0 \Rightarrow \alpha = 0 \ \lor\ \alpha = -1$$

↳ отпада, јер $x_n > 0$

$$\Rightarrow \lim_{n\to\infty} x_n = 0.$$

$$x_{n+1} = \frac{x_n}{1+x_n+x_n^2}$$

$$\lim_{n\to\infty}(n \cdot x_n) = \lim_{n\to\infty} \frac{n}{\frac{1}{x_n}} = \lim_{n\to\infty} \frac{(n+1)-n}{\frac{1}{x_{n+1}} - \frac{1}{x_n}} = \lim_{n\to\infty} \frac{1}{\frac{x_n^2+x_n+1}{x_n} - \frac{1}{x_n}} = \lim_{n\to\infty} \frac{x_n}{x_n^2+x_n} = \lim_{n\to\infty} \frac{1}{x_n+1} = 1.$$

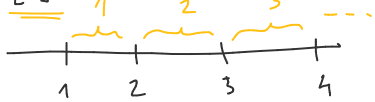
$x_n \searrow \Rightarrow \frac{1}{x_n} \nearrow$

$x_n \to 0 \Rightarrow \frac{1}{x_n} \to \infty$

} услови за Штолца!

(4) Доказати да за $n \in \mathbb{N}$ важи $S = \int_1^{n+1} \frac{\sin(\pi x)}{[x]}\,dx = \frac{2}{\pi} \cdot \sum_{k=1}^{n} \frac{(-1)^k}{k}$.

$[x]$?



$$\int_1^{n+1} = \int_1^2 + \int_2^3 + \int_3^4 + \ldots + \int_n^{n+1} = \sum_{k=1}^{n} \int_k^{k+1} \frac{\sin(\pi x)}{[x]}\,dx$$

$$\cos k\pi = (-1)^k$$

$$\int_k^{k+1} \frac{\sin(\pi x)}{[x]}\,dx = \int_k^{k+1} \frac{\sin(\pi x)}{k}\,dx = \frac{1}{k} \cdot \frac{1}{\pi}\left(-\cos(\pi x)\right)\Big|_k^{k+1} = \frac{1}{k\pi}\left(\cos(k\pi) - \cos((k+1)\pi)\right) = \frac{(-1)^k - (-1)^{k+1}}{k\pi}$$

$$= \frac{(-1)^k}{k\pi} \cdot \left(1 - (-1)\right) = \frac{2(-1)^k}{k\pi}$$

$[x] = k$, $x \in [k, k+1)$

ово не важи за $x = k+1$, али то је само 1 тачка и интеграл не зависи од ње (мере је нула).

$$\Rightarrow S = \sum_{k=1}^{n} \int_k^{k+1} \ldots = \sum_{k=1}^{n} \frac{2(-1)^k}{k\pi} = \frac{2}{\pi} \sum_{k=1}^{n} \frac{(-1)^k}{k}$$

$$\int^{n+1} \frac{\sin(\pi x)}{\ } dx = \frac{2}{\ } \sum^{n} \frac{(-1)^k}{\ } \quad \Big/ \lim$$

$$\int_1^{n+1} \frac{\sin(\pi x)}{[x]}\,dx = \frac{2}{\pi}\sum_{k=1}^{n} \frac{(-1)^k}{k} \quad \Big/ \lim_{n\to\infty}$$

$$\sum \frac{(-1)^k}{k} \text{ konv.} \Rightarrow \exists \lim_{n} \sum^{n} \frac{(-1)^k}{k}$$

$$\Rightarrow \quad \int_1^{+\infty} \frac{\sin(\pi x)}{[x]}\,dx = \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \Rightarrow \int_1^{+\infty} \frac{\sin(\pi x)}{[x]}\,dx \text{ konv.}$$