Петак 16.04.2021.

(1) $\sum \frac{n!}{n^n}$

$$a_n = \frac{n!}{n^n} \qquad \frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \frac{\cancel{(n+1)!}}{\cancel{n!}} \cdot \frac{n^n}{(n+1)^{n+\cancel{1}}} = \left(\frac{n}{n+1}\right)^n = \underbrace{\left(1-\frac{1}{n+1}\right)^{-(n+1)}}_{\to e}{}^{\overbrace{\frac{n}{-(n+1)}}^{\to -1}} \xrightarrow{n\to\infty} e^{-1} = \frac{1}{e} < 1$$

$\underset{(\text{Даламбер})}{\Longrightarrow}$ конв.

Стирлингова формула: $n! \sim \sqrt{2n\pi}\cdot\left(\frac{n}{e}\right)^n, \ n\to\infty \ (n\in\mathbb{N})$; $\lim\limits_{n\to\infty} \frac{n!}{\sqrt{2n\pi}\left(\frac{n}{e}\right)^n} = 1$

(2) $\sum \frac{n!\,e^n}{n^{n+p}}$, $p\in\mathbb{R}$

$$\frac{n!\,e^n}{n^{n+p}} \sim \sqrt{2n\pi}\cdot\left(\frac{n}{e}\right)^n\cdot\frac{e^n}{n^n\cdot n^p} = \sqrt{2\pi}\cdot\frac{n^{1/2}}{n^p} = \sqrt{2\pi}\cdot\frac{1}{n^{p-\frac{1}{2}}}$$

$$\sum\frac{n!\,e^n}{n^{n+p}} < \infty \iff \sum\sqrt{2\pi}\cdot\frac{1}{n^{p-\frac{1}{2}}} < \infty \iff p-\frac{1}{2} > 1 \iff p > \frac{3}{2}$$

(3) Испитати конвергенцију низа $x_n = 1+\frac{1}{\sqrt2}+\frac{1}{\sqrt3}+\dots+\frac{1}{\sqrt n}-2\sqrt n$.

$$x_{n+1} = x_n + \underbrace{\frac{1}{\sqrt{n+1}}+2\sqrt n-2\sqrt{n+1}}_{a_n} \qquad , x_1 = 1-2=-1 \ , \ a_0 = x_1$$

$$x_{n+1} = x_n + a_n = x_{n-1}+a_{n-1}+a_n = x_{n-2}+a_{n-2}+a_{n-1}+a_n = \dots = \overset{a_0}{\underset{\Pi}{x_1}}+a_1+a_2+\dots+a_n = \sum_{k=0}^{n} a_n$$

$$\exists \lim x_{n+1} \iff \exists \lim \sum_{k=0}^{n} a_n \iff \sum a_n \text{ конв.}$$

$$a_n = \frac{1}{\sqrt{n+1}}+2\sqrt n-2\sqrt{n+1} = \frac{1}{\sqrt{n+1}}+2\sqrt n\left(1-\sqrt{1+\frac1n}\right) \sim \frac{1}{\sqrt{n+1}}+2\sqrt n\left(1-\left(1+\frac{1}{2n}-\frac{1}{8n^2}+o\left(\frac{1}{n^2}\right)\right)\right) \sim$$

$$\sqrt{1+\frac1n} = \left(1+\frac1n\right)^{1/2} \sim 1+\frac{1}{2n}-\frac{1}{8n^2}+o\left(\frac{1}{n^2}\right)$$

$$\binom{1/2}{2} = -\frac18$$

$$\frac{1}{\sqrt{n+1}} = \frac{1}{\sqrt n}\left(1+\frac1n\right)^{-1/2} = \frac{1}{\sqrt n}\left(1-\frac{1}{2n}+o\left(\frac1n\right)\right)$$

$$\sim \frac{1}{\sqrt n}\left(1-\frac{1}{2n}+o\left(\frac1n\right)\right)+2\sqrt n\left(-\frac{1}{2n}+\frac{1}{8n^2}+o\left(\frac{1}{n^2}\right)\right) =$$

$$= \frac{1}{\sqrt n}-\frac{1}{2n^{3/2}}+o\left(\frac{1}{n^{3/2}}\right)-\frac{1}{\sqrt n}+\frac{1}{4n^{3/2}}+o\left(\frac{1}{n^{3/2}}\right) =$$

$$= -\frac{1}{4n^{3/2}}+o\left(\frac{1}{n^{3/2}}\right), \ n\to\infty$$

овај ред сталног знака (<0) почевши од нект. члана

$$\frac{-1}{\sqrt{n+1}} = \frac{1}{\sqrt{n}}\left(1+\frac{1}{n}\right)^{-1/2} = \frac{1}{\sqrt{n}}\left(1-\frac{1}{2n}+o\left(\frac{1}{n}\right)\right)$$

овај ред сталног знака (<0) почевши од неког члана

$$\sum \frac{-1}{4n^{3/2}} \text{ конв} \overset{\text{ПК}}{\Rightarrow} \sum a_n \text{ конв.} \Rightarrow x_n \text{ конв.}$$

3 Рабеов критеријум

(а) $\exists \alpha>1$ т.д. $\forall n\geq n_0$

$$n\left(\frac{a_n}{a_{n+1}}-1\right)\geq \alpha \Rightarrow \sum a_n<\infty$$

$$n\left(\frac{a_n}{a_{n+1}}-1\right)\leq 1 \Rightarrow \sum a_n=\infty$$

(б) $\lim\limits_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right)=l$

$l>1 \Rightarrow \sum a_n<\infty$

$l<1 \Rightarrow \sum a_n=\infty$

$l=1$ не знамо

4 Гаусов критеријум

$$\frac{a_n}{a_{n+1}} = \lambda + \frac{\mu}{n} + O\left(\frac{1}{n^{1+\delta}}\right), \text{ за неко } \delta>0, \quad n\to\infty$$

$\sum a_n<\infty$ **акко** $\lambda>1 \lor (\lambda=1 \land \mu>1)$

Напомена: 1° $\lambda>1$ и $\lambda<1$ су Даламбер

2° $\lambda=1$, $\mu>1$ и $\mu<1$ су Рабе

3° $\lambda=1$, $\mu=1$ (Бертранов критеријум)

Пр. $\sum \frac{1}{n^p}$:

$$\frac{a_n}{a_{n+1}} = \frac{(n+1)^p}{n^p} = \left(1+\frac{1}{n}\right)^p = 1+\frac{p}{n}+\underbrace{\boxed{\binom{p}{2}\frac{1}{n^2}+o\left(\frac{1}{n^2}\right)}}_{o\left(\frac{1}{n^{3/2}}\right)} = 1+\frac{p}{n}+o\left(\frac{1}{n^{3/2}}\right) = 1+\frac{p}{n}+O\left(\frac{1}{n^{3/2}}\right), \quad n\to\infty$$

Гаус: $\sum \frac{1}{n^p}<\infty \Leftrightarrow p>1$.

(4) $$\sum \underbrace{\frac{(2n-1)!!}{(2n)!!}\cdot\frac{1}{2n+1}}_{a_n}$$

$n! = n\cdot(n-1)\cdot\ldots\cdot 1$

$n!! = n\cdot(n-2)\cdot(n-4)\cdot\ldots$

$(2n+1)!! = (2n+1)(2n-1)\cdot\ldots\cdot 3\cdot 1$

$(2n)!! = (2n)(2n-2)\cdot\ldots\cdot 4\cdot 2$

$\frac{(2n-1)!!}{}\cdot\frac{1}{}$

$2n+2$

a_n

$(2n+1)!! = (2n+1)(2n-1)\cdot \ldots \cdot 3 \cdot 1$

$(2n)!! = (2n)(2n-2)\cdot \ldots \cdot 4 \cdot 2$

$n!! = n\cdot(n-2)!!$

$$\frac{a_n}{a_{n+1}} = \frac{\dfrac{(2n-1)!!}{(2n)!!}\cdot\dfrac{1}{2n+1}}{\dfrac{(2n+1)!!}{(2n+2)!!}\cdot\dfrac{1}{2n+3}} = \frac{\cancel{(2n-1)!!}}{\underbrace{\cancel{(2n+1)!!}}_{2n+1}}\cdot\frac{\overbrace{\cancel{(2n+2)!!}}^{2n+2}}{\cancel{(2n)!!}}\cdot\frac{2n+3}{2n+1} =$$

$$= \frac{(2n+2)(2n+3)}{(2n+1)^2}$$

$$n\cdot\left(\frac{a_n}{a_{n+1}}-1\right) = n\cdot\left(\frac{4n^2+10n+6}{4n^2+4n+1}-1\right) = n\cdot\frac{6n+5}{4n^2+4n+1} = \frac{6n^2+5n}{4n^2+4n+1} \xrightarrow[n\to\infty]{} \frac{3}{2} > 1$$

$$\overset{\text{Рабе}}{\Longrightarrow} \sum a_n < \infty$$

(5) $\displaystyle\sum \underbrace{\frac{p(p+1)\cdot\ldots\cdot(p+n-1)}{n!}\cdot\frac{1}{n^q}}_{a_n}$, $p>0$

$p>0$ није неопходно:

нпр. $p=-\frac{5}{2}$: $n=1$: $-\frac{5}{2}$

$n=2$: $\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)$

$n=3$: $\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)$

$n=4$: $\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\cdot\frac{1}{2}$

$n=5$: $\left(-\frac{5}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)\cdot\frac{1}{2}\cdot\frac{3}{2}$

ови су сталног знака (почевши од неког)

$a_n > 0$

$$\frac{a_n}{a_{n+1}} = \frac{\dfrac{\cancel{p(p+1)\ldots(p+n-1)}}{\cancel{n!}}\cdot\dfrac{1}{n^q}}{\dfrac{\cancel{p(p+1)\ldots(p+n-1)}(p+n)}{\underbrace{\cancel{(n+1)!}}_{n+1}}\cdot\dfrac{1}{(n+1)^q}} =$$

$$= \frac{(n+1)^q}{n^q}\cdot\frac{1}{\frac{p+n}{n+1}} = \left(1+\frac{1}{n}\right)^q\cdot\frac{n+1}{n+p} \overset{\text{Lin}}{=}$$

$$= \left(1+\frac{1}{n}\right)^q\cdot\frac{1+\frac{1}{n}}{1+\frac{p}{n}} = \left(1+\frac{1}{n}\right)^{q+1}\cdot\left(1+\frac{p}{n}\right)^{-1} = \left(1+\frac{q+1}{n}+\binom{q+1}{2}\frac{1}{n^2}+o\left(\frac{1}{n^2}\right)\right)\left(1-\frac{p}{n}+\frac{3}{2}\cdot\frac{p^2}{n^2}+o\left(\frac{1}{n^2}\right)\right) =$$

$$= 1-\frac{p}{n}+\underbrace{\frac{3}{2}\frac{p^2}{n^2}+o\left(\frac{1}{n^2}\right)+\frac{q+1}{n}-\frac{p(q+1)}{n^2}+\binom{q+1}{2}\frac{1}{n^2}}_{=o\left(\frac{1}{n^{3/2}}\right)} = 1+\frac{q-p+1}{n}+O\left(\frac{1}{n^{3/2}}\right)$$

Даље: конв $\Leftrightarrow$ $q-p+1>1 \Leftrightarrow q>p$.

$\boxed{7}$ (Кошијев кондензациони критеријум) $a_n>0$ и $a_n\downarrow$

$$\sum_n a_n<\infty \Leftrightarrow \sum_k 2^k\cdot a_{2^k}$$

$\log(2^k) = k\cdot\log 2$

пр. $\sum \frac{1}{n(\log n)^q} < \infty \iff \sum_k 2^k \frac{1}{2^k \cdot (\log 2^k)^q} < \infty \iff \sum_k \frac{1}{(\log 2)^q \cdot k^q} < \infty \iff q > 1$

⑥ $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \cdot (\log n)^p \cdot (\log(\log n))^q}$

$$\sum \frac{1}{n(\log n)^p \cdot (\log(\log n))^q} < \infty \iff \sum 2^n \cdot \frac{1}{2^n (\log 2^n)^p (\log(\log 2^n))^q} < \infty$$

$$\iff \sum \frac{1}{n^p \cdot (\log 2)^p \cdot (\log(n \cdot \log 2))^q} < \infty$$

$$\iff \sum \frac{1}{n^p (\log 2)^p \cdot (\log n + \log\log 2)^q} < \infty \quad \sim \frac{1}{n^p (\log 2)^p (\log n)^q}$$

$$\iff \sum \frac{1}{n^p \underbrace{(\log 2)^p}_{\text{const}} (\log n)^q} < \infty$$

$$\iff \sum \frac{1}{n^p (\log n)^q} < \infty \iff p > 1 \ \vee \ (p = 1 \ \wedge \ q > 1)$$

Редове с произволни членове

Деф. $\sum a_n$ абсолютно конвергира (АК) ако $\sum |a_n| < \infty$.

Важи: АК $\Rightarrow$ конвергира ($\sum a_n$ див. $\Rightarrow \sum |a_n|$ див.)

обратното не важи! (пример: $\sum \frac{(-1)^n}{n}$)

абс. \ обикн.	К	Д
К	АПС. КВ.	
Д	УСЛОВНА КВ.	ДИВ.

(условна, неапсолутна, ...)

$\boxed{1}$ (Лајбниц) $\sum (-1)^n \cdot a_n$ и $a_n \searrow 0 \Rightarrow$ онда конв.

$$\sum (-1)^n \cdot a_n = -a_1 + a_2 - a_3 + a_4 - \dots = -a_1 + (a_2 - a_3) + (a_4 - a_5) + \dots$$

(7) Доказати да је $\sum \frac{(-1)^n}{n}$ условно конвергентан.

1) $\sum \frac{(-1)^n}{n}$ конв.

$$\underbrace{(-1)^n \cdot \frac{1}{n}}_{a_n}, \quad \frac{1}{n} \searrow 0 \underset{\text{Лајбниц}}{\Rightarrow} \sum \frac{(-1)^n}{n} \text{ конв.}$$

$\sum (-1)^n \rightrightarrows \begin{matrix} 1 \\ -1 \end{matrix}$ ×

$$\left| \sum_{n=1}^{N} (-1)^n \right| \leq 1$$

2) $\sum \left| \frac{(-1)^n}{n} \right|$ не конв.

$$\left| \frac{(-1)^n}{n} \right| = \frac{1}{n}, \quad \sum \frac{1}{n} \text{ див.}$$

$\Rightarrow$ условно конв.