Понедељак 15.03.2021.

Тејлорова формула

$f:(a,b)\to\mathbb{R}$, $x_0\in(a,b)$

f је n-пута диф. у x_0

$$f(x) = \underbrace{f(x_0) + \frac{f'(x_0)}{1!}\cdot(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{\text{Тејлоров. пол. степена } n} + o((x-x_0)^n),\quad x\to x_0$$

Често $x_0=0$ (Маклоренова)

Пр. $f(x)=e^x$, $x_0=0$ $\quad f'(x)=\dots=f^{(n)}(x)=e^x$

$$f(0)=f'(0)=\dots=f^{(n)}(0)=1$$

$$e^x=f(x)=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\dots+\frac{f^{(n)}(0)}{n!}x^n+o(x^n)=\sum_{k=0}^{n}\frac{x^k}{k!}+o(x^n),\ \boxed{x\to 0}$$

обавезно

$x^2=x^3+o(x)$, $x\to0$

$\Longleftrightarrow$

$x^2-x^3=o(x)$, $x\to0$

$\Longleftrightarrow$

$\frac{x^2-x^3}{x}=o(1)$, $x\to0$

$\frac{x^2-x^3}{x}\to0$, $x\to0$

$f=o(g)$, $x\to x_0$

$\frac{f}{g}\to0$, $x\to x_0$

Развоји:

- $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^n}{n!}+o(x^n)$, $x\to0$
- $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots+(-1)^n\frac{x^{2n+1}}{(2n+1)!}+o(x^{2n+1})$, $x\to0$ — може и $o(x^{2n+2})$
- $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots+(-1)^n\frac{x^{2n}}{(2n)!}+o(x^{2n})$, $x\to0$ — може и $o(x^{2n+1})$
- $(1+x)^\alpha = 1+\binom{\alpha}{1}x+\binom{\alpha}{2}x^2+\dots+\binom{\alpha}{n}x^n+o(x^n)$, $x\to0$

  $$\binom{\alpha}{k}=\frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-k+1)}{k!},\quad \alpha\in\mathbb{R},\ k\in\mathbb{N}_0$$
- $\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots+(-1)^{n+1}\frac{x^n}{n}+o(x^n)$, $x\to0$

Нпр: $n=1$, $e^x=1+x+o(x)\Rightarrow e^x-1=x+o(x)\Rightarrow\frac{e^x-1}{x}=1+o(1)\Rightarrow\lim_{x\to0}\frac{e^x-1}{x}=1$.

$(1+x)^\alpha=1+\binom{\alpha}{1}x+o(x)\Rightarrow(1+x)^\alpha=1+\alpha x+o(x)\Rightarrow\frac{(1+x)^\alpha-1}{x}=\alpha+o(1)\Rightarrow\lim_{x\to0}\frac{(1+x)^\alpha-1}{x}=\alpha$.

$\binom{\alpha}{1}=\frac{\alpha}{1}=\alpha$

$$(1+x)^{\alpha}=1+\binom{\alpha}{1}x+o(x) \Rightarrow (1+x)^{\alpha}=1+\alpha x+o(x) \Rightarrow \frac{(1+x)^{\alpha}-1}{x}=\alpha+o(1) \Rightarrow \lim_{x\to 0}\frac{(1+x)^{\alpha}-1}{x}=\alpha.$$

$$\binom{\alpha}{1}=\frac{\alpha}{1}=\alpha$$

① Развити у околини нуле у Т. ред.

а) $f(x)=\dfrac{1}{1+x}$

б) $f(x)=\operatorname{arctg} x$

а) $$f(x)=\frac{1}{1+x}=(1+x)^{-1}=1+\binom{-1}{1}x+\binom{-1}{2}x^2+\dots+\binom{-1}{n}x^n+o(x^n)=\sum_{k=0}^{n}\binom{-1}{k}x^k+o(x^n)=\sum_{k=0}^{n}(-1)^k x^k+o(x^n),\ x\to 0$$

$$=1-x+x^2-x^3+\dots+(-1)^n x^n+o(x^n),\ x\to 0$$

$(1+x)^{\alpha},\ \alpha=-1$

$$\binom{-1}{k}=\frac{(-1)(-1-1)(-1-2)\dots(-1-k+1)}{k!}=\frac{(-1)(-2)(-3)\dots(-k)}{k!}=\frac{(-1)^k\cdot\cancel{1\cdot 2\cdot\dots\cdot k}}{\cancel{k!}}=(-1)^k$$

б) $$\operatorname{arctg} x=\sum_{k=0}^{n}\frac{(\operatorname{arctg})^{(k)}(0)}{k!}\cdot x^k+o(x^n)=\sum_{k=0}^{N}\frac{(\operatorname{arctg})^{(2k+1)}(0)}{(2k+1)!}+o(x^n)=\frac{x}{1}-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\dots+\frac{(-1)^N}{2N+1}x^N+o(x^{N+1}),\ x\to 0$$

Са пр. часа: $$\boxed{\begin{aligned} &f^{(2k)}(0)=0 \quad ,\forall k\in\mathbb{N}_0\\ &f^{(2k+1)}(0)=(-1)^k\cdot(2k)! \quad ,\forall k\in\mathbb{N}_0\end{aligned}}$$

$$\frac{(-1)^k(2k)!}{(2k+1)!}=\frac{(-1)^k}{2k+1}$$

N-највећи непаран $N\le n$. $N\in\{n,n-1\}$

② Развити у околини нуле:

а) $f(x)=\operatorname{tg} x$, до петог степена

б) $f(x)=e^{\sin x}\cdot\cos x$, до трећег степена ← домаћи

а) $$f(x)=\operatorname{tg} x=\frac{\sin x}{\cos x}=\sin x\cdot(\cos x)^{-1}=\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+o(x^6)\right)\cdot\left(1-\frac{x^2}{2}+\frac{x^4}{4!}+o(x^5)\right)^{-1}=$$

оба бар до 5. ст.

$(1+t)^{\alpha},\ \alpha=-1,\ t=-\frac{x^2}{2}+\frac{x^4}{4}+o(x^5)$

$$=\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+o(x^6)\right)\cdot\left(1-\left(-\frac{x^2}{2}+\frac{x^4}{4!}+o(x^5)\right)+\left(-\frac{x^2}{2}+\frac{x^4}{4!}+o(x^5)\right)^2-\left(-\frac{x^2}{2}+\frac{x^4}{4!}+o(x^5)\right)^3+\underbrace{o\left(\left(-\frac{x^2}{2}+\frac{x^4}{4!}+o(x^5)\right)^3\right)}_{=o(x^6)}\right)$$

$$\binom{-1}{k}=(-1)^k \rightsquigarrow (1+t)^{-1}=1-t+t^2-t^3+t^4-t^5+o(t^5),\ t\to 0$$

t је 2. ст. ⇒ довољно до трећег по t да развијемо

$$=(\ \dots\ )\cdot\left(1+\frac{x^2}{2}-\frac{x^4}{4!}+o(x^5)+\underbrace{\left(-\frac{x^2}{2}\right)^2}_{=\frac{x^4}{4}}+0\right)=$$

↳ од тога нама ништа

↳ једини степена ≤ 5

$4!=24$

$x\cdot o(x^5)=o(x^6)$

$3!=6$

$5!=120$

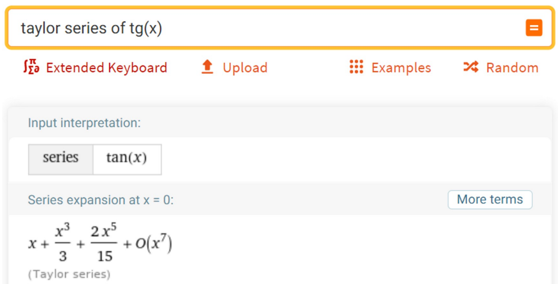
$\left(-\frac{x^3}{3!}\right)\cdot\frac{5}{24}x^4=o(x^6)$

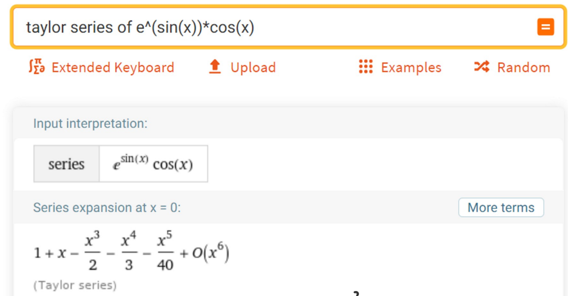
$$= \left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+o(x^6)\right)\cdot\left(1+\frac{x^2}{2}+\frac{5}{24}x^4+o(x^5)\right)=$$

$$= x+\frac{x^3}{2}+\frac{5}{24}x^5+o(x^6)-\frac{x^3}{3!}-\frac{x^5}{12}+\frac{x^5}{5!}=$$

$$= x+\frac{x^3}{3}+\frac{2}{15}x^5+o(x^6)$$

$$\frac{5}{24}-\frac{1}{12}+\frac{1}{120}=\frac{25-10+1}{120}=\frac{16}{120}=\frac{2}{15}$$





до трећег: $1+x-\frac{x^3}{2}+o(x^3)$

③ Нађи константе $a,b,c\in\mathbb{R}$ тд. важи $f(x)=ax+b+\frac{c}{x}+o\left(\frac{1}{x}\right)$, $x\to+\infty$

готово

а) $f(x)=\frac{(x+1)^3}{(x-1)^2}$

б) $f(x)=x\cdot\sqrt[3]{1+\frac{2}{x}}$

в) $f(x)=\sqrt{x^2+1}+\log\frac{\sqrt{x^2+1}-1}{x}$

$x\to+\infty$? $\rightsquigarrow$ $\frac{1}{x}\longrightarrow 0$, $t=\frac{1}{x}$ користимо оно што знамо

$t\to 0$ $(1+t)^{1/2}$

в) $$\sqrt{x^2+1}=(x^2+1)^{1/2}=\left(x^2\left(1+\frac{1}{x^2}\right)\right)^{1/2}=x\cdot\left(1+\frac{1}{x^2}\right)^{1/2}=x\cdot\left(1+\binom{1/2}{1}\frac{1}{x^2}+o\left(\frac{1}{x^2}\right)\right)=x\cdot\left(1+\frac{1}{2x^2}+o\left(\frac{1}{x^2}\right)\right)=$$

$x^2\to\infty$

$(x^2)^{1/2}=|x|=x$, $x>0$ јер гледамо $x\to\infty$

$$=x+\frac{1}{2x}+o\left(\frac{1}{x}\right),\ x\to\infty$$

$$\frac{\sqrt{x^2+1}-1}{x}=\frac{1}{x}\cdot\left(x+\frac{1}{2x}+o\left(\frac{1}{x}\right)-1\right)=1-\frac{1}{x}+\frac{1}{2x^2}+o\left(\frac{1}{x^2}\right),\ x\to\infty$$

$o\left(\frac{1}{x}\right)$

$$\log\frac{\sqrt{x^2+1}-1}{x}=\log\Big(1\underbrace{-\frac{1}{x}+\frac{1}{2x^2}+o\left(\frac{1}{x^2}\right)}_{t\to 0}\Big)=t+o(t)=-\frac{1}{x}+\frac{1}{2x^2}+o\left(\frac{1}{x^2}\right)+o\left(-\frac{1}{x}+\frac{1}{2x^2}+o\left(\frac{1}{x^2}\right)\right)$$

до 1. ст.

$$=-\frac{1}{x}+o\left(\frac{1}{x}\right)$$

$$o\left(-\frac{1}{x}+\frac{1}{2x^2}+o\left(\frac{1}{x^2}\right)\right)=o\left(-\frac{1}{x}\right)+o\left(\frac{1}{2x^2}\right)+o\left(o\left(\frac{1}{x^2}\right)\right)=o\left(\frac{1}{x}\right)+o\left(\frac{1}{x^2}\right)+o\left(\frac{1}{x^2}\right)=o\left(\frac{1}{x}\right),\ x\to\infty$$

$$o\left(-\frac{1}{x}+\frac{1}{2x^2}+o\left(\frac{1}{x^2}\right)\right) = o\left(-\frac{1}{x}\right)+o\left(\frac{1}{2x^2}\right)+\underbrace{o\left(o\left(\frac{1}{x^2}\right)\right)}_{o(o(f))=o(f)} = o\left(\frac{1}{x}\right)+\underbrace{o\left(\frac{1}{x^2}\right)+o\left(\frac{1}{x^2}\right)}_{o\left(\frac{1}{x^2}\right)} = o\left(\frac{1}{x}\right),\ x\to\infty$$

$$f(x) = x+\frac{1}{2x}+o\left(\frac{1}{x}\right)+\left(-\frac{1}{x}\right)+o\left(\frac{1}{x}\right) = x-\frac{1}{2x}+o\left(\frac{1}{x}\right)$$

$$a=1,\ b=0,\ c=-\frac{1}{2}$$

4) Израхувати ліміти:

а) $\lim\limits_{x\to 0}\dfrac{\operatorname{ch} x-\cos x}{x^2}$

$\operatorname{ch} x = \dfrac{e^x+e^{-x}}{2}$ – косинус гіперболічний

$\operatorname{sh} x = \dfrac{e^x-e^{-x}}{2}$ – синус гіпер.

б) $\lim\limits_{x\to 0}\dfrac{\operatorname{tg} x - x}{x-\sin x}$

в) $\lim\limits_{x\to 0}\dfrac{1-(\cos x)^{\sin x}}{x^3}$

домати

г) $\lim\limits_{x\to+\infty} x^{3/2}\left(\sqrt{x+1}+\sqrt{x-1}-2\sqrt{x}\right)$

можна и $o(x^3)$

а) $$\lim_{x\to 0}\frac{\frac{1}{2}(e^x+e^{-x})-\cos x}{x^2} = \lim_{x\to 0}\frac{\frac{1}{2}\left(1+\cancel{x}+\frac{x^2}{2!}+o(x^2)+1+(\cancel{-x})+\frac{(-x)^2}{2!}+o(x^2)\right)-\left(1-\frac{x^2}{2}+o(x^2)\right)}{x^2} =$$

$$=\lim_{x\to 0}\frac{\frac{1}{2}\left(\cancel{2}+x^2+o(x^2)\right)-\cancel{1}+\frac{x^2}{2}-o(x^2)}{x^2} =$$

$$=\lim_{x\to 0}\frac{\frac{x^2}{2}+o(x^2)+\frac{x^2}{2}}{x^2} =$$

$$=\lim_{x\to 0}\frac{x^2+o(x^2)}{x^2} = \lim_{x\to 0}\left(1+o(1)\right)=1, \qquad \lim_{x\to 0} o(1)=0$$

г) $$\lim_{x\to\infty} x^{3/2}\left(\sqrt{x+1}+\sqrt{x-1}-2\sqrt{x}\right) \underset{t=\frac{1}{x}}{=} \lim_{t\to 0+}\frac{1}{t^{3/2}}\cdot\left(\sqrt{\frac{1}{t}+1}+\sqrt{\frac{1}{t}-1}-2\sqrt{\frac{1}{t}}\right) = S$$

бер до 2 ст. бо $\frac{3}{2}+\frac{1}{2}=2$ $\quad \frac{1}{t^{3/2}}\cdot\frac{1}{\sqrt{t}}=\frac{1}{t^2}$

$$\sqrt{\frac{1}{t}+1} = \sqrt{\frac{1}{t}}\cdot\sqrt{1+t} = \frac{1}{\sqrt{t}}(1+t)^{1/2} = \frac{1}{t^{1/2}}\cdot\left(1+\frac{1}{2}t-\frac{1}{8}t^2+o(t^2)\right)$$

$$\binom{1/2}{1} = \frac{1}{2} \quad , \binom{1/2}{2} = \frac{(1/2)(1/2-1)}{2!} = \frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)}{2} = -\frac{1}{8}$$

$$\sqrt{\frac{1}{t}-1} = \sqrt{\frac{1}{t}}\left(1-t\right)^{1/2} = \frac{1}{t^{1/2}}\cdot\left(1-\frac{1}{2}t-\frac{1}{8}t^2+o(t^2)\right)$$

$$S = \lim_{t\to 0+} \frac{1}{t^{3/2}}\left(\frac{1}{t^{1/2}}\cdot\left(1+\frac{1}{2}t-\frac{1}{8}t^2+o(t^2)\right)+\frac{1}{t^{1/2}}\cdot\left(1-\frac{1}{2}t-\frac{1}{8}t^2+o(t^2)\right)-\frac{2}{t^{1/2}}\right) =$$

$$= \lim_{t\to 0+} \frac{1}{t^2}\left(0+0t-\frac{1}{4}t^2+o(t^2)\right) = \lim_{t\to 0+}\left(-\frac{1}{4}+o(1)\right) = -\frac{1}{4}$$