Петак 14.05.2021.

Теореме о средњој вредности за интеграле

$\boxed{T_1}$ (Прва теорема о средњој вредности)

прочитати T45 у професорским предавањима или T2 из професорове скрипте

$f,g:[a,b]\to\mathbb{R}$, g интеграбилна, f непрекидна, $g\geq 0$ (или $g\leq 0$)

$$\Rightarrow \exists \xi\in[a,b],\quad \int_a^b f(x)\cdot g(x)\,dx = f(\xi)\cdot\int_a^b g(x)\,dx$$

$\boxed{T_2}$ (Друга теорема о средњој вредности)

прочитати T47 у професорским предавањима или T5 из професорове скрипте

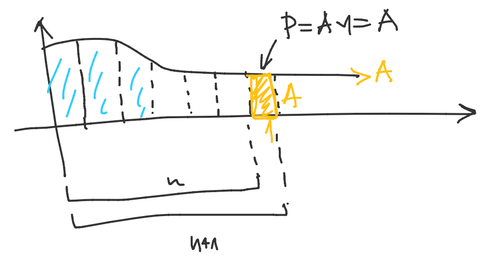
$f,g:[a,b]\to\mathbb{R}$, f непрекидна, g је C^1 и монотона

$$\Rightarrow \exists \xi\in[a,b],\quad \int_a^b f(x)\cdot g(x)\,dx = g(a)\cdot\int_a^{\xi} f(x)\,dx + g(b)\cdot\int_{\xi}^b f(x)\,dx$$

① $f\in C(\mathbb{R})$ и $\lim\limits_{x\to\infty} f(x)=A>0$. Наћи $\lim\limits_{n\to\infty}\int_0^1 f(nx)\,dx$.

$$\int_0^1 f(nx)\,dx \underset{\substack{nx=t\\ n\,dx=dt}}{=} \int_0^n f(t)\cdot\frac{dt}{n} = \frac{1}{n}\int_0^n f(t)\,dt$$

x	0	1
t	0	n



$$a_n=\int_0^n f(t)\,dt$$

$$b_n=n$$

$$\lim_{n\to\infty}\int_0^1 f(nx)\,dx = \lim_{n\to\infty}\frac{\int_0^n f(t)\,dt}{n} = \lim_{n\to\infty}\frac{a_n}{b_n} \underset{\substack{\text{штолц}\\ (b_n\nearrow\infty)}}{=} \lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty}\frac{\int_0^{n+1}-\int_0^n}{1} = \lim_{n\to\infty}\int_n^{n+1} f(t)\,dt$$

$$\to \int_n^{n+1} f(t)\,dt = \int_n^{n+1} f(t)\cdot 1\,dt \overset{\exists\xi\in[n,n+1]}{\underset{\substack{T_1:\\ f \text{ непр.}\\ 1 \text{ инт.}\quad 1\geq 0}}{=}} f(\xi)\cdot\underbrace{\int_n^{n+1} 1\,dt}_{1} = f(\xi)$$

$$\xi=\xi(n)\in[n,n+1]$$

$$\lim_{n\to\infty}\int_0^1 f(nx)\,dx = \lim_{n\to\infty}\int_n^{n+1} f(t)\,dt = \lim_{n\to\infty} f(\xi(n)) = f\left(\lim_{n\to\infty}\xi(n)\right) = A.$$

$$\xi(n) \xrightarrow[n\to\infty]{} \infty$$

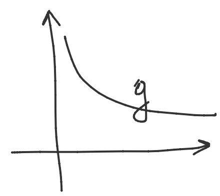
② Наћи $\displaystyle\lim_{n\to\infty}\int_n^{n+p}\frac{\sin x}{x}\,dx$, $p>0$.

I начин: $\boxed{T_2}$

$f(x)=\sin x$

$g(x)=\frac{1}{x}$

- f непр ✓
- g је C^1 ✓ (на $[n,n+p]$)
- g монотона? $\quad g'(x) = -\frac{1}{x^2} < 0$



$T_2 \Rightarrow \exists\, \xi(n)\in[n,n+p]$ т.д. $\qquad \int \sin x = -\cos x + c$

$$\int_n^{n+p}\frac{\sin x}{x}\,dx = g(n)\cdot\int_n^{\xi}\sin x\,dx + g(n+p)\cdot\int_{\xi}^{n+p}\sin x\,dx =$$

$$= \underbrace{\frac{1}{n}}_{\to 0}\underbrace{\left(\cos(n)-\cos(\xi)\right)}_{\text{ограничени!}} + \underbrace{\frac{1}{n+p}}_{\to 0}\cdot\underbrace{\left(\cos(\xi)-\cos(n+p)\right)}_{\text{ограничени!}}$$

$$\lim_{n\to\infty}\int_n^{n+p}\frac{\sin x}{x}\,dx = \lim_{n\to\infty}\frac{\overbrace{\cos n-\cos\xi}^{\text{огр!}}}{\underbrace{n}_{\to\infty}} + \lim_{n\to\infty}\frac{\cos\xi-\cos(n+p)}{n+p} = 0+0=0.$$

$\qquad\qquad\downarrow 0 \qquad\qquad \downarrow 0$

II начин: (директно процењујемо)

$$0\le \left|\int_n^{n+p}\frac{\sin x}{x}\,dx\right| \le \int_n^{n+p}\left|\frac{\sin x}{x}\right|dx \overset{|\sin x|\le 1}{\underset{(x>0)}{\le}} \int_n^{n+p}\frac{1}{x}\,dx = \log x\Big|_n^{n+p} = \log(n+p)-\log n = \log\left(\frac{n+p}{n}\right) = \log\left(1+\frac{p}{n}\right)$$

$\to 0$ $\qquad$ нej. Δ за интеграле $\qquad$ $\to 0$ $\quad \log 1 = 0$

lim: $\left|\int_n^{n+p}\frac{\sin x}{x}\,dx\right| \to 0 \Rightarrow \int_n^{n+p}\frac{\sin x}{x}\,dx \longrightarrow 0$

$\lim\limits_{n\to\infty}$: $\quad \left|\int_n^{n+p} \frac{\sin x}{x}\,dx\right| \to 0 \Rightarrow \int_n^{n+p} \frac{\sin x}{x}\,dx \xrightarrow[n\to\infty]{} 0$

(3) Нека је $h(x) = \int_x^{2x} \frac{\sin^n t}{t^{n+1}}\,dt$. Наћи $\lim\limits_{x\to 0+} h(x)$.

$n\in\mathbb{N}$

$\frac{\sin t}{t} \to 1,\ t\to 0+$

T_1 примењујемо: $f(t) = \left(\frac{\sin t}{t}\right)^n$

$g(t) = \frac{1}{t}$

- f непр? $\quad t>0$: јер $t\in[x,2x],\ x>0$
- $g(t)\geq 0$
- g интегр? $\Leftarrow g$ непр.

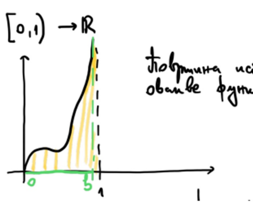
$$\Rightarrow \exists \xi(x)\in[x,2x] \text{ т.д. } \int_x^{2x} \frac{\sin^n t}{t^{n+1}}\,dt = f(\xi(x))\cdot\int_x^{2x} g(t)\,dt = \left(\frac{\sin \xi(x)}{\xi(x)}\right)^n \cdot \underbrace{\int_x^{2x}\frac{dt}{t}}_{\log 2} \xrightarrow[x\to 0+]{} 1^n\cdot\log 2 = \log 2.$$

$$\int_x^{2x}\frac{dt}{t} = \log t\,\Big|_x^{2x} = \log(2x) - \log x = \log\left(\frac{2x}{x}\right) = \log 2.$$

$$\underset{\xi(x)\in[x,2x]}{x\to 0+ \Rightarrow} \xi(x)\to 0+ \Rightarrow \frac{\sin\xi(x)}{\xi(x)} \xrightarrow[x\to 0+]{} 1$$

Несвојствени интеграл

$f: [0,1) \to \mathbb{R}$



Површина испод графика овакве функције?

пр.
- $\int_0^{+\infty} \frac{1}{1+x^2}\,dx = \operatorname{arctg} x\,\Big|_0^{+\infty} = \underbrace{\lim\limits_{x\to\infty}\operatorname{arctg} x}_{=\frac{\pi}{2}} - \underbrace{\operatorname{arctg} 0}_{=0} = \frac{\pi}{2}$
- $\int_0^1 \frac{dx}{\sqrt{x}}$?
- $\int_{-1}^1 \frac{dx}{x}$? $\quad 0\in[-1,1]$

Деф: • $f : [a, \beta) \to \mathbb{C}$, $a < \beta \le +\infty$

• $f \in \mathcal{R}[a, b]$, на сваком затвореном и ограниченом интервалу $[a, b] \subseteq [a, \beta)$

Риман-интегр. ф-је на $[a,b]$

Ако $\exists \lim\limits_{b \to \beta} \int_a^b f(x)\,dx$ онда се овај лимес назива **НЕСВОЈСТВЕНИМ ИНТЕГРАЛОМ** ф-је f и означава са

$$\int_a^\beta f(x)\,dx := \lim_{b \to \beta} \int_a^b f(x)\,dx$$

β - сингуларитет несвојственог интеграла

/ $\beta \in \mathbb{R}$ $\quad \lim\limits_{b \to \beta -}$

$\beta = +\infty$ $\quad \lim\limits_{b \to +\infty}$ /

Ако $\nexists \lim\limits_{b \to \beta} \int_a^\beta f(x)\,dx$ онда кажемо да **НЕСВОЈСТВЕНИ ИНТЕГРАЛ ДИВЕРГИРА.**

Тврђење: (линеарност)

$f, g : [a, \beta) \to \mathbb{C}$

$f, g \in \mathcal{R}[a, b]$, $\forall [a, b] \subseteq [a, \beta)$

Ако несвојств. инт.

$\int_a^\beta f\,dx$ и $\int_a^\beta g\,dx$

конв. тада конвергира и $\int_a^\beta (\lambda f + \mu g)\,dx$

и важи

$$\int_a^\beta (\lambda f + \mu g)\,dx = \lambda \int_a^\beta f\,dx + \mu \int_a^\beta g\,dx,$$

при чему су $\lambda, \mu \in \mathbb{C}$.

Слично важи за $f, g : (\alpha, \beta] \to \mathbb{C}$.

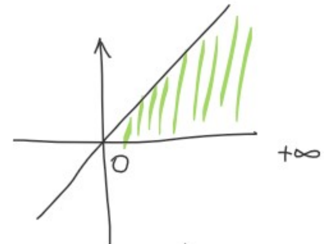
→ Слично и за доњу границу

(некад се пише и $f \in \mathcal{R}_{loc}[a, \beta)$) локално интеграбилна

• $f \in \mathcal{R}_{loc}[a, c)$ и $f \in loc(c, b] \Rightarrow \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$

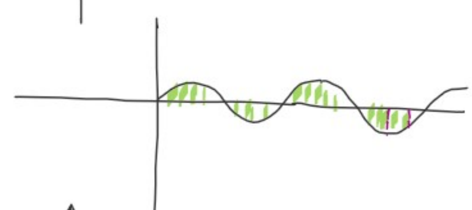
• $f \in \mathcal{R}_{loc}[a, b)$ и $a < c < b \Rightarrow f \in \mathcal{R}_{loc}[c, b)$

пр. 1) $\int_0^{+\infty} x\,dx = \lim\limits_{M \to \infty} \int_0^M x\,dx = \lim\limits_{M \to \infty} \frac{M^2}{2} = \infty \Rightarrow$ дивергира



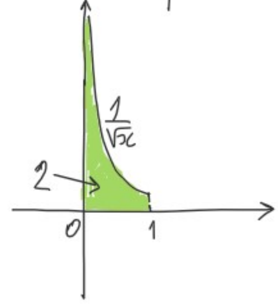
2) $\int_0^{+\infty} \sin x\,dx = \lim\limits_{M \to \infty} \int_0^M \sin x\,dx = \lim\limits_{M \to \infty} \left(-\cos x\right)\Big|_0^M = \lim\limits_{M \to \infty} \left(1 - \cos M\right)$

$\nexists \lim \Rightarrow$ дивергира



3) $\int_0^1 \frac{dx}{\sqrt{x}} = \lim\limits_{\varepsilon \to 0+} \int_\varepsilon^1 \frac{dx}{\sqrt{x}} = \lim\limits_{\varepsilon \to 0+} \left(2\sqrt{x}\right)\Big|_\varepsilon^1 = \lim\limits_{\varepsilon \to 0+} \left(2 - 2\sqrt{\varepsilon}\right) = 2.$

$\Rightarrow$ конвергира



(1) $\int_0^1 \log x\,dx = \lim\limits_{\varepsilon \to 0+} \int_\varepsilon^1 \log x\,dx = \lim\limits_{\varepsilon \to 0+} \left(x \log x - x\right)\Big|_\varepsilon^1 =$

парцијална

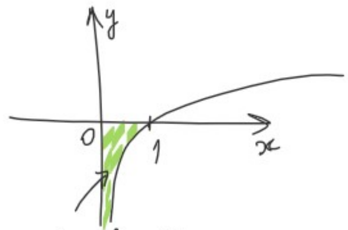
$\left(x \log x - x\right)' = x \cdot \frac{1}{x} + 1 \cdot \log x - 1 =$
$= \log x$

$= -1 - \lim (\varepsilon \log \varepsilon - \varepsilon) = -1$

$-\log x$

парцијална

$$= -1 - \lim_{\varepsilon\to0+}\left(\underbrace{\varepsilon\log\varepsilon}_{\to0} - \underbrace{\varepsilon}_{\to0}\right) = -1$$



ова површина је $=1$.

$$\lim_{\varepsilon\to0+}\varepsilon\log\varepsilon = \lim_{\varepsilon\to0}\frac{\log\varepsilon}{\frac{1}{\varepsilon}} \overset{?}{=} \lim_{\varepsilon\to0+}\frac{\frac{1}{\varepsilon}}{-\frac{1}{\varepsilon^2}} = -\lim_{\varepsilon\to0+}\varepsilon = 0$$

Лопитал

(2) $\displaystyle\int_1^{+\infty}\frac{dx}{x(x+1)}$

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \quad /\cdot x(x+1)$$

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

$$1 = A(x+1) + Bx$$

$$\left.\begin{array}{l} A+B=0 \\ A=1 \end{array}\right\} A=-B=1.$$

НЕ СМЕ: $\displaystyle\int_1^{+\infty}\frac{dx}{x(x+1)} = \boxed{\int_1^{+\infty}\frac{dx}{x}} - \boxed{\int_1^{+\infty}\frac{dx}{1+x}} = \infty - \infty = ?$

дивергирају!

$$\int_1^{+\infty}\frac{dx}{x} = \lim_{M\to\infty}\log x\Big|_1^M = \lim_{M\to\infty}\log M = +\infty$$

$$\frac{1}{x(x+1)}, \frac{1}{x}, \frac{1}{x+1} \in C([1,M])$$

одређени инт.

$$\int_1^{+\infty}\frac{dx}{x(x+1)} = \lim_{M\to\infty}\boxed{\int_1^M\frac{dx}{x(x+1)}} = \lim_{M\to\infty}\left(\int_1^M\frac{dx}{x} - \int_1^M\frac{dx}{x+1}\right) =$$

$$= \lim_{M\to\infty}\left(\log|x|\Big|_1^M - \log|x+1|\Big|_1^M\right) = \lim_{M\to\infty}\left(\log M - \underbrace{\log 1}_{=0} - \log(M+1) + \log 2\right) =$$

$$= \lim_{M\to\infty}\log\underbrace{\frac{2M}{M+1}}_{\to2} = \log 2.$$

(3) $\displaystyle\int_1^{+\infty}\frac{dx}{x(x+1)(x+2)} = \lim_{M\to\infty}\int_1^M\frac{dx}{x(x+1)(x+2)}$

$$\frac{1}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2} \quad \ldots \quad A=C=\frac{1}{2}, \quad B=-1$$

домаћи: довршити $\;\; --- \lim_{M\to+\infty} \log \underbrace{\frac{\sqrt{M(M+2)}}{M+1}}_{\to 1} + \log 2 - \frac{1}{2}\log 3 = \log\frac{2}{\sqrt{3}}.$

НАПОМЕНА:

да је било: $\displaystyle\int_{-2}^{+\infty} \frac{dx}{x(x+1)(x+2)} \overset{?}{=} \int_{-2}^{-1} \frac{dx}{x(x+1)(x+2)} + \int_{-1}^{0} \frac{dx}{x(x+1)(x+2)} + \int_{0}^{+\infty} \frac{dx}{x(x+1)(x+2)}$

$$\to \lim_{\varepsilon\to-2+} \lim_{M\to-1-} \int_{\varepsilon}^{M} \frac{dx}{x(x+1)(x+2)}$$ (мора овај поновљени lim да $\exists$)

• Важе Т о смени и парцијалној, само треба водити рачуна да све конвергира.

(4) $$\int_{2}^{+\infty} \frac{1}{\sqrt{x}} \log\left(\frac{x+1}{x-1}\right) dx = \left| \begin{array}{ll} u=\log\left(\frac{x+1}{x-1}\right) & dv=\frac{dx}{\sqrt{x}} \\ du = \left(\log(x+1)-\log(x-1)\right)' dx = & v=2\sqrt{x} \\ = \left(\frac{1}{x+1} - \frac{1}{x-1}\right)dx = & \\ = \frac{2\,dx}{(x+1)(x-1)} & \end{array} \right| =$$

$$= \log\left(\frac{x+1}{x-1}\right)\cdot 2\sqrt{x}\,\Big|_{2}^{+\infty} - 4\int_{2}^{+\infty} \frac{\sqrt{x}\,dx}{(x+1)(x-1)}$$

под условом да ово десно $\exists$

$$\log\left(\frac{x+1}{x-1}\right)\cdot 2\sqrt{x}\,\Big|_{2}^{+\infty} = \lim_{x\to+\infty}\left(\log\left(\frac{x+1}{x-1}\right)\cdot 2\sqrt{x}\right) - 2\sqrt{2}\log 3 = -2\sqrt{3}\log 3$$

$$\log\left(1+\frac{2}{x-1}\right)2\sqrt{x} \sim \frac{2}{x-1}\cdot 2\sqrt{x} \sim \frac{4\sqrt{x}}{x-1} \xrightarrow[x\to+\infty]{} 0$$

$$\int_{2}^{+\infty} \frac{\sqrt{x}\,dx}{(x+1)(x-1)} = \left| \begin{array}{l} \sqrt{x}=t \\ t^2=x \\ 2t\,dt=dx \end{array} \quad \begin{array}{c|c|c} x & 2 & +\infty \\ \hline t & \sqrt{2} & +\infty \end{array} \right|$$

$$= \int_{\sqrt{2}}^{+\infty} \frac{t\cdot 2t\,dt}{(t^2+1)(t-1)(t+1)} = \lim_{M\to\infty} 2\cdot\int_{\sqrt{2}}^{M} \frac{t^2\,dt}{(t^2+1)(t-1)(t+1)} = \ldots$$ (испитати да конв.)

под условом

под услови да овој дел $\exists$

$\sqrt{2} \quad (t^2+1)(t-1)(t+1) \qquad M \to \infty \quad \sqrt{2} \quad (t^2+1)(t-1)(t+1)$

домашно: довршите

$$\frac{t^2}{(t^2+1)(t-1)(t+1)} = \frac{A}{t-1} + \frac{B}{t+1} + \frac{Ct+D}{t^2+1}$$ итд.