Четвртак 13.05.2021.

(1) $$\int_{-1}^{1} \frac{dx}{x^2-2x\cos\alpha+1} = I, \quad 0<\alpha<\pi$$

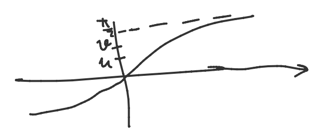
$$x^2-2x\cos\alpha+1 = (x-\cos\alpha)^2-\cos^2\alpha+1 = (x-\cos\alpha)^2+\sin^2\alpha$$

$$\frac{x-\cos\alpha}{\sin\alpha} = t \qquad dt = \frac{dx}{\sin\alpha}$$

x	-1	1
t	$\frac{-1-\cos}{\sin}$	$\frac{1-\cos}{\sin}$

$$I = \int_{\frac{-1-\cos\alpha}{\sin\alpha}}^{\frac{1-\cos\alpha}{\sin\alpha}} \frac{\sin\alpha\, dt}{\sin^2\alpha\,(t^2+1)} = \frac{1}{\sin\alpha}\int_{\dots}^{\dots}\frac{dt}{t^2+1} = \frac{1}{\sin\alpha}\cdot\Big(\underbrace{\operatorname{arctg}\Big(\overbrace{\frac{1-\cos\alpha}{\sin\alpha}}^{>0}\Big)}_{u} + \underbrace{\operatorname{arctg}\Big(\overbrace{\frac{1+\cos\alpha}{\sin\alpha}}^{>0}\Big)}_{v}\Big) = \frac{\pi}{2\sin\alpha}.$$

$$u, v \in \left(0, \frac{\pi}{2}\right)$$



$$\operatorname{tg} u = \frac{1-\cos\alpha}{\sin\alpha}, \quad \operatorname{tg} v = \frac{1+\cos\alpha}{\sin\alpha}$$

$$\operatorname{tg} u \cdot \operatorname{tg} v = \frac{(1-\cos\alpha)(1+\cos\alpha)}{\sin^2\alpha} = \frac{1-\cos^2\alpha}{\sin^2\alpha} = 1$$

$$\Rightarrow \operatorname{tg} u = \operatorname{ctg} v = \operatorname{tg}\left(\frac{\pi}{2}-v\right) \Rightarrow u = \frac{\pi}{2}-v \Rightarrow u+v = \frac{\pi}{2}$$

(2) $f(x) = x\left(\frac{\pi}{2} - \operatorname{arctg}\frac{1}{x}\right)$

а) f има непр. инверзну фју

б) $\int_0^{\pi/4} f^{-1}(y)\,dy$

а) f непр?

$$\lim_{x\to 0+}\Big(\underbrace{x\cdot\frac{\pi}{2}}_{\to 0} - \underbrace{x}_{\to 0}\,\underbrace{\operatorname{arctg}\frac{1}{x}}_{\to\frac{\pi}{2}}\Big) = 0 \qquad \lim_{x\to 0-}\Big(\underbrace{x\cdot\frac{\pi}{2}}_{\to 0} - \underbrace{x}_{\to 0}\,\underbrace{\operatorname{arctg}\frac{1}{x}}_{\to-\frac{\pi}{2}}\Big) = 0$$

$$\tilde{f}(x) = \begin{cases} f(x), & x\neq 0 \\ 0, & x=0 \end{cases} \qquad \text{непр.}$$

derivative of x(pi/2-arctg(1/x))

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Derivative: Approximate form

$$\frac{d}{dx}\left(x\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{1}{x}\right)\right)\right)=\frac{x}{x^2+1}-\tan^{-1}\left(\frac{1}{x}\right)+\frac{\pi}{2}=f'(x)$$

second derivative of x(pi/2-arctg(1/x))

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Derivative:

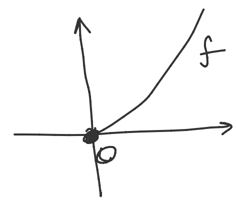
$$\frac{d^2}{dx^2}\left(x\left(\frac{\pi}{2}-\tan^{-1}\left(\frac{1}{x}\right)\right)\right)=\frac{2}{(x^2+1)^2}=f''(x)$$

$$f''(x)>0 \Rightarrow f'(x)\nearrow$$

$$\lim_{x\to 0+} f'(x) = 0-\frac{\pi}{2}+\frac{\pi}{2}=\underline{\underline{0}}$$

$$\Rightarrow f'(x)>0 \quad , x>0$$

$$\Rightarrow f(x)\nearrow \quad , f(0)=0 \Rightarrow f(x)>0 , x>0$$



f непр. и $f\nearrow \Rightarrow \exists f^{-1}$ и непр.

б) $\displaystyle\int_0^{\pi/4} f^{-1}(y)\,dy = \Big/ \begin{matrix} y=f(x) \\ dy=f'(x)\,dx \end{matrix} \Big/$ ← смемо јер је f диј. $\exists f^{-1}$

$f(1)=1\cdot\left(\frac{\pi}{2}-\frac{\pi}{4}\right)=\frac{\pi}{4}$

y	0	$\pi/4$
x	0	1

$$=\int_0^1 \underbrace{f^{-1}(f(x))}_{x}\cdot f'(x)\,dx = \int_0^1\left(\frac{x^2}{x^2+1}+\frac{\pi}{2}x - x\,\mathrm{arctg}\,\frac{1}{x}\right)dx = \ldots = \frac{\pi}{4}+\frac{1}{2}$$

← доказати

лако

парцијална!

доказати: $\displaystyle\int_0^{2/3}\left(x-\frac{x^3}{3}\right)\cdot\frac{dx}{(1-x^2)\cdot\sqrt{1-2x^2}} = \ldots = \frac{1}{9}+\frac{\pi}{6}-\frac{2}{3}\,\mathrm{arctg}\,\frac{1}{3}$

увести смену: $t=\sqrt{1-2x^2}$

(или $t=1-2x^2$, $u=\sqrt{t}$)

Основна теорема интегралног рачуна

$\boxed{\text{Т}}$ • Ако је f интеградибилна на $[a,b]$, онда је

$F(x)=\int_a^x f(t)\,dt$ непрекидна на $[a,b]$

• Ако је f непрекидна у тачки $x_0\in(a,b)$, онда је

$F(x)=\int_a^x f(t)\,dt$ диференцијабилна у x_0 и важи $F'(x_0)=f(x_0)$.

Напомене: • уместо a смо могли да узмемо и неку другу тачку

• иако се интеграли по t, у граници је x, па зато F зависи само од x

f непр $\Rightarrow F$ диф

f интегр. $\Rightarrow F$ непр.

пр. $F(x)=\int_0^{x^2} e^{-t^2}\,dt \rightsquigarrow F'(x)=?$

$G(x)=\int_0^{x} e^{-t^2}\,dt \rightsquigarrow G'(x)=e^{-x^2}$

(*) $F(x)=\int_a^{\varphi(x)} f(t)\,dt$, φ диф.

$F'(x)=?$

$G(x)=\int_a^{x} f(t)\,dt$, $G(\boxed{\varphi(x)})=\int_a^{\boxed{\varphi(x)}} f(t)\,dt = F(x)$

$\Downarrow$

$G'=f$ $\quad\Rightarrow F=G\circ\varphi$

$$F'=G'(\varphi)\cdot\varphi' \Rightarrow F'(x)=G'(\varphi(x))\cdot\varphi'(x) \underset{G'=f}{=} f(\varphi(x))\cdot\varphi'(x)$$

пр. $\left(\int_a^{\overset{\varphi}{x^2}} \overset{f}{e^{-t^2}}\,dt\right)'_x = e^{-(x^2)^2}\cdot 2x = e^{-x^4}\cdot 2x.$

(*) домаћи: $F(x)=\int_{\psi(x)}^{\varphi(x)} f(t)\,dt$, $F'(x)=?$

$$= \int_{\psi(x)}^{a} + \int_a^{\varphi(x)} = -\int_a^{\psi(x)} + \int_a^{\varphi(x)}$$

3) $f(x)=e^{x^2}\cdot\int_0^x e^{-t^2}\,dt - x$. Доказати да је f строго растућа на $\mathbb{R}$.

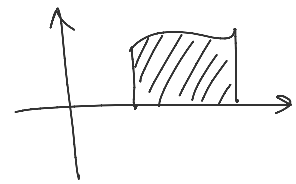
e^{-t^2} непр. $\Rightarrow \int_0^x e^{-t^2}\,dt$ диф. $\Rightarrow f$ диф.

$$f'(x)=(e^{x^2})'\cdot\int_0^x e^{-t^2}\,dt + e^{x^2}\cdot\left(\int_0^x e^{-t^2}\,dt\right)' - 1 = 2x\cdot e^{x^2}\cdot\int_0^x e^{-t^2}\,dt + \underbrace{e^{x^2}\cdot e^{-x^2}}_{\cancel{1}} - \cancel{1} =$$

$$= 2x\cdot e^{x^2}\cdot\int_0^x e^{-t^2}\,dt$$

$f'(x) \geq 0$?

1° $x>0$, $e^{-t^2} \geq 0 \Rightarrow \int_0^x e^{-t^2}\,dt > 0$



$2x>0$

$e^{x^2}>0$

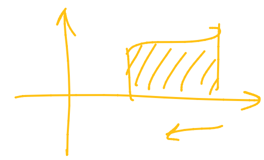
$\Rightarrow f'(x)>0$

2° $x=0$

$f'(0)=0$

3° $x<0$, $e^{-t^2} \geq 0 \Rightarrow \int_0^x e^{-t^2}\,dt < 0$

у супр. поретку



$2x<0$

$e^{x^2}>0$

$\Rightarrow f'(x)>0$

$\Rightarrow$ Следи да је $f \nearrow$.

(4) а) $\lim\limits_{x\to 0+} \dfrac{\int_0^x \cos(t^2)\,dt}{x}$ б) $\lim\limits_{x\to +\infty} \dfrac{\left(\int_0^x e^{t^2}\,dt\right)^2}{\int_0^x e^{2t^2}\,dt}$ в) $\lim\limits_{x\to \infty} \dfrac{\left(\int_0^x (\operatorname{arctg} t)^2\,dt\right)^2}{x^2+1}$

домаћи (2 Лопитала)

а) $\lim\limits_{x\to 0+} \int_0^x \cos(t^2)\,dt$

$\cos(t^2)$ непр. $\Rightarrow \underbrace{\int_0^x \cos(t^2)\,dt}_{=F(x)}$ диф. , $F'(x) = \cos(x^2)$

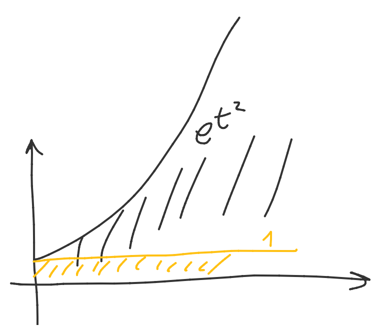
Напомена:
$\int_0^x \cos(t^2)\,dt$
$\int_0^x e^{t^2}\,dt$ итд.
нису елементарне

$\lim\limits_{x\to 0+} \dfrac{F'(x)}{(x)'} = \lim\limits_{x\to 0+} \dfrac{\cos(x^2)}{1} = 1 \Rightarrow \lim\limits_{x\to 0+} \dfrac{F(x)}{x} = 1.$

Лопитал

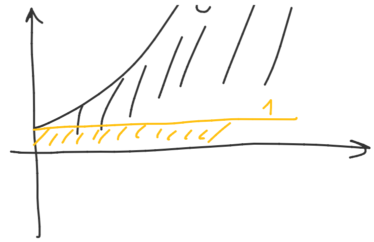
б) $F(x) = \int_0^x e^{t^2}\,dt$

$G(x) = \int_0^x e^{2t^2}\,dt$



б) $F(x) = \int\limits_0 e^{t^2}dt$

$G(x) = \int\limits_0^x e^{2t^2}dt$



$x \to \infty:$ $\left.\begin{matrix} e^{t^2} \geqslant 1 \\ e^{2t^2} \geqslant 1 \end{matrix}\right\} \Rightarrow F(x) \geqslant \int\limits_0^x 1\,dt = x \to \infty$

$G(x) \geqslant \int\limits_0^x 1\,dt = x \to \infty$

$$\lim_{x\to\infty} \frac{(F^2(x))'}{G'(x)} = \lim_{x\to\infty} \frac{2F(x)F'(x)}{G'(x)} = \lim_{x\to\infty} \frac{2\cdot F(x)\cdot e^{x^2}}{e^{2x^2}} = \lim_{x\to\infty} \frac{2\cdot \overbrace{\int_0^x e^{t^2}dt}^{F(x)\,\to\infty}}{\underbrace{e^{x^2}}_{\to\infty}} \quad (*)$$

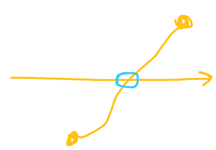
$$\lim_{x\to\infty} \frac{(2\cdot F(x))'}{(e^{x^2})'} = \lim_{x\to\infty} \frac{2F'(x)}{e^{x^2}\cdot 2x} = \lim_{x\to\infty} \frac{e^{x^2}}{e^{x^2}\cdot x} = \lim_{x\to\infty} \frac{1}{x} = 0$$

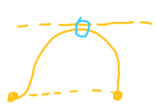
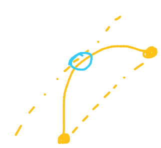
$\exists \Rightarrow \exists \lim (*) \Rightarrow \exists$ исходный lim

Лопиталь Лопиталь

$$\lim_{x\to\infty} \frac{F^2(x)}{G(x)} = \lim_{x\to\infty} \frac{(F^2(x))'}{G(x)} = \lim_{x\to\infty} \frac{2F'(x)}{(e^{x^2})'} = 0.$$

(5) $f: [a,b] \to \mathbb{R}$ непр. и $\int\limits_a^b f(x) = 0$. Доказать $\exists c \in (a,b)$ т.ч. $f(c) = \int\limits_a^c f(x)dx$.

1)  (Т о промежуточном значении)

2)   (Ролль, Лагранж)

$F(x) = \int\limits_a^x f(t)dt$, $F'(x) = f(x) \overset{?}{=} F(x)$ → да ли $\exists c$ т.ч. $F'(c) = F(c)$.

Удобно $G(x) = F(x)\cdot e^{-x}$

$G'(x) = F'(x)\cdot e^{-x} + F(x)\cdot(-e^{-x}) = e^{-x}(F'(x) - F(x))$

$\text{''}0 \Rightarrow F'(x) = F(x)$

да ли $\exists c$ т.д. $G'(c) = 0$?

Ролова Т: $G(a) = F(a) \cdot e^{-a} = \underbrace{\int_a^a f(t)\,dt}_{0} \cdot e^{-a} = 0$

$$G(b) = F(b) \cdot e^{-b} = \underbrace{\int_a^b f(t)\,dt}_{0} \cdot e^{-b} = 0$$

$G(a) = G(b) \Rightarrow \exists c \in (a,b)$ т.д. $G'(c) = 0 \Rightarrow F'(c) = F(c)$

$$f(c) = \int_a^c f(t)\,dt.$$

⑥ Нати све непр. ф-је $f : (0,+\infty) \to (0,+\infty)$ т.д. $\forall x > 0$ важи

$$2x \cdot \int_0^x f(t)\,dt = f(x) \quad (*) \quad /' \quad \Rightarrow \int_0^x f(t)\,dt = \frac{f(x)}{2x}$$

f непр. $\Rightarrow F(x) = \int_0^x f(t)\,dt$ диф. и $F'(x) = f(x)$.

$$2x \cdot F'(x) + 2 \cdot F(x) = f'(x)$$

$$2x \cdot f(x) + 2 \cdot \boxed{\int_0^x f(t)\,dt} = f'(x)$$

знамо да елиминишемо!

1) још 1 извод $\to f''$ ...

1) изразимо из $(*)$

$$\Rightarrow 2x f(x) + 2 \cdot \frac{f(x)}{2x} = f'(x) \quad / \cdot x$$

$$2x^2 f(x) + f(x) = x \cdot f'(x)$$

$$f(x) \cdot (2x^2 + 1) = x \cdot f'(x) \quad / : x$$

$$f'(x) - f(x) \cdot \left(2x + \frac{1}{x}\right) = 0$$

идеја : нађи $G(x)$ тд. $G'(x)$

нпр. $G(x) = f(x) \cdot e^{g(x)}$ ← покушати у овом облику

$\vdots$ (доказати)

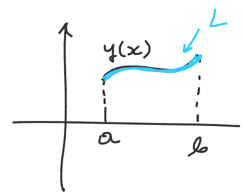
може се узети $g(x) = -x^2 - \log x$

$G(x) = f(x) \cdot e^{-x^2 - \log x} \Rightarrow G'(x) = 0 \therefore$

решење: $f(x) = c \cdot e^{x^2 + \log x}$, $c \in \mathbb{R}^+$.

(7) Нађи дужину криве $y = \int_0^x \boxed{\sqrt{\cos(2t)}}$, $0 \le x \le \frac{\pi}{4}$.

дужина криве: $$L = \int_a^b \sqrt{1 + (y')^2}\, dx$$



$$y' = \left(\int_0^x \sqrt{\cos(2t)}\, dt \right)' = \sqrt{\cos(2x)}$$

$[a, b] = [0, \frac{\pi}{4}]$

$$L = \int_0^{\pi/4} \sqrt{1 + (y')^2}\, dx = \int_0^{\pi/4} \sqrt{1 + \left(\sqrt{\cos(2x)}\right)^2}\, dx = \int_0^{\pi/4} \sqrt{\underbrace{1 + \cos(2x)}_{2\cos^2 x}}\, dx = \int_0^{\pi/4} \sqrt{2} \cdot \cos(x)\, dx =$$

$\cos x \ge 0$

$$= \sqrt{2} \cdot (\sin x) \Big|_0^{\pi/4} = \sqrt{2} \left(\frac{1}{\sqrt{2}} - 0 \right) = 1.$$