Понедељак 12.10.2020.

$$\limsup A_n = \bigcap_{n=1}^{\infty} \boxed{\bigcup_{k=n}^{\infty} A_k} = \underbrace{(A_1 \cup A_2 \cup \dots)} \cap (A_2 \cup A_3 \cup \dots) \cap (A_3 \cup A_4 \cup \dots) \cap \dots$$

$$x \in \limsup A_n \iff (\forall n \in \mathbb{N})\ x \in \bigcup_{k=n}^{\infty} A_k$$

$n=1:\ x \in A_1 \cup A_2 \cup \dots \Rightarrow x \in A_{n_1}$

$n=n_1+1:\ x \in A_{n_1+1} \cup A_{n_1+2} \cup \dots \Rightarrow x \in A_{n_2},\ n_2 > n_1$

$n=n_2+1:\ x \in A_{n_2+1} \cup \dots \Rightarrow x \in A_{n_3},\ n_3 > n_2$

$\vdots$

$\iff$ x се налази у бесконачно много од њих

$$\liminf A_n \subseteq \limsup A_n$$

8. $(A_n)_{n\in\mathbb{N}}$ и A, B

$$A_n = \begin{cases} A, & 2 \mid n \\ B, & 2 \nmid n \end{cases} \qquad \begin{matrix} A_1 & A_2 & A_3 & A_4 & \dots \\ B & A & B & A & \end{matrix}$$

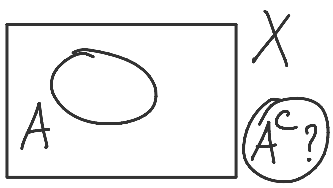
Нађи $\liminf A_n$ и $\limsup A_n$.

$$\liminf A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} (A_n \cap A_{n+1} \cap A_{n+2} \cap \dots) = \bigcup_{n=1}^{\infty} (A \cap B) = A \cap B$$

$$\limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \bigcap_{n=1}^{\infty} (A_n \cup A_{n+1} \cup A_{n+2} \cup \dots) = \bigcap_{n=1}^{\infty} (A \cup B) = A \cup B$$

$A \in \mathcal{P}(X)$ $\qquad$ $\underline{C_X A} = A^c = X \setminus A$

$(A \subseteq X)$



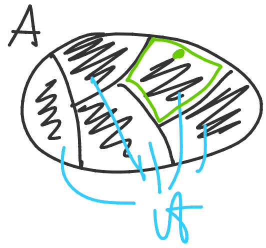
1. $C_X C_X A = A$

$$C_X(\underbrace{C_X A}) = \{x \in X \mid \overbrace{x \notin C_X A}^{\neg(x \in C_X A)}\} = \{x \in X \mid \neg(x \notin A)\} =$$

$$= \{x \in X \mid \neg(\neg(x \in A))\} = \{x \in X \mid x \in A\} = A$$

A

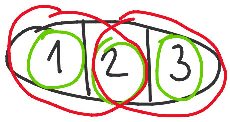
$= \{x \in A \mid \neg(\exists!(\ldots$



② строго разлагање A

$(\forall x\in A)(\exists! X\in \mathcal{A})\ x\in X$

$A=\{1,2,3\}$



$\mathcal{A}_1=\{\{1\},\{2\},\{3\}\}$

$\mathcal{A}_2=\{\{1,2\},\{2,3\}\}$ $\longrightarrow$ не важи $(\forall x\in A)(\exists! X\in\mathcal{A})\ x\in X$, $\{1,2\}\cap\{2,3\} = \{2\}\neq\emptyset$

$\mathcal{A}_2$ није строго разл.

$$\left.\begin{array}{l} \{1\}\cup\{2\}\cup\{3\} = \bigcup \mathcal{A}_1 = A \\ \{1\}\cap\{2\}=\emptyset \\ \{1\}\cap\{3\}=\emptyset \\ \{2\}\cap\{3\}=\emptyset \end{array}\right\}$$ јесте ст. раз.

③ $(a,b)=\{\{a\},\{a,b\}\}$

$(a,b)\neq\{a,b\}$

$(a,b)\neq(b,a)$

$\{a,b\}=\{b,a\}$

$(a,b)=(u,v) \Leftrightarrow a=u \wedge b=v$

$\boxed{\Leftarrow}$
$$\left.\begin{array}{l} a=u \Rightarrow \{a\}=\{u\} \\ b=v \Rightarrow \{a,b\}=\{u,v\} \end{array}\right\} \Rightarrow \{\{a\},\{a,b\}\}=\{\{u\},\{u,v\}\} \Rightarrow (a,b)=(u,v)$$

$\boxed{\Rightarrow}$ $(a,b)=(u,v)$

$\{\{a\},\{a,b\}\}=\{\{u\},\{u,v\}\}$

$\{a\}\neq\{a,b\}$, осим ако $a=b$ 1° $\{\{a\},\{a,b\}\}=\{\{a\},\{a,a\}\}=$
$=\{\{a\},\{a\}\}=\{\{a\}\}$

$\Rightarrow \{u\}=\{u,v\} \Rightarrow u=v$

аналогно

2° $a\neq b$ $\{a\},\{a,b\}$ 2.1° ако $\{a\}=\{u,v\} \Rightarrow u=v=a$

$\{\{u\},\{u,v\}\} = \{\{a\}\} \Rightarrow a=b$

2° $a \neq b$

$\{a\}, \{a,b\}$ 2.1° ако $\{a\} = \{u, v\} \Rightarrow u = v = a$

$\{u\}, \{u,v\}$

$\{\{u\}, \{u,v\}\} = \{\{a\}\} \Rightarrow a = b$ ↯

2.2° $\{a\} = \{u\} \Rightarrow a = u$

$\{a,b\} = \{u,v\} \Rightarrow b = v$ ✓

$$A \times B = \{ z \in \underline{\mathcal{P}(\mathcal{P}(A \cup B))} \mid (\exists a \in A)(\exists b \in B)\ z = (a,b) \}$$

$\hookrightarrow \{\{a\}, \{a,b\}\}$

$\{a\}, \{a,b\} \subseteq A \cup B$
$\in \mathcal{P}(A \cup B)$

$(a,b) = \{\{a\}, \{a,b\}\} \in \mathcal{P}(\mathcal{P}(A \cup B))$

④ Домаћи: Показати да за скупове X и Y важи

а) $X \times Y = \emptyset \iff X = \emptyset \vee Y = \emptyset$

б) $X \times Y \subseteq A \times B \iff X \subseteq A \wedge Y \subseteq B$

в) $(X \times A) \cup (X \times B) = X \times (A \cup B)$

г) $(X \times A) \cap (Y \times B) = (X \cap Y) \times (A \cap B)$

Profesorkin sajt: http://poincare.matf.bg.ac.rs/~jovanadj/

а) $X \times Y = \emptyset \iff X = \emptyset \vee Y = \emptyset$

$\boxed{\Rightarrow}$ $X \times Y = \emptyset \Rightarrow X = \emptyset \vee Y = \emptyset$

ппс (претпоставимо супротно) $\neg(X = \emptyset \vee Y = \emptyset)$

$\neg(X = \emptyset) \wedge \neg(Y = \emptyset)$

$X \neq \emptyset \wedge Y \neq \emptyset$

$\Downarrow$

треба да добијемо $\neg(X \times Y = \emptyset)$

$X \times Y \neq \emptyset$

Доказ помоћу супротне претпоставке

$p \Rightarrow q$

$p \Rightarrow p_1 \Rightarrow p_2 \Rightarrow \dots \Rightarrow q \rightarrow$ директно

$\neg q \Rightarrow q_1 \Rightarrow \dots \Rightarrow \neg p$ } → суп. претп.

$p \Rightarrow q \iff \neg q \Rightarrow \neg p$

контрадикција: ↯

$$\left.\begin{array}{l} X \neq \emptyset \Rightarrow \exists x \in X \\ Y \neq \emptyset \Rightarrow \exists y \in Y \end{array}\right\} (x,y) \in X \times Y \Rightarrow X \times Y \neq \emptyset \ ↯ \Rightarrow \text{важи } (\Rightarrow)$$

$\boxed{\Leftarrow}$ ппс $X \times Y \neq \emptyset \Rightarrow (x,y) \in X \times Y \Rightarrow \{\{x\}, \{x,y\}\} \in X \times Y$

$\Rightarrow \exists z \in \mathcal{P}(\mathcal{P}(X \cup Y)),\ z = \{\{x\}, \{x,y\}\}$

$x \in X \Rightarrow X \neq \emptyset$

$y \in Y \Rightarrow Y \neq \emptyset$ ↯

$\Rightarrow$ други смер (**)

(*), (**) важи еквиваленција

в) $(X\times A)\cup(X\times B) = X\times(A\cup B)$

$\boxed{\Rightarrow}$ $(u,v)\in(X\times A)\cup(X\times B) \Leftrightarrow (u,v)\in X\times A \vee (u,v)\in X\times B$

$\Leftrightarrow (u\in X \wedge v\in A) \vee (u\in X \wedge v\in B)$

$\Leftrightarrow (u\in X \vee u\in X)\wedge(u\in X\vee v\in B)\wedge(v\in A\vee u\in X)\wedge(v\in A\vee v\in B)$

$\Leftrightarrow u\in X \wedge \underbrace{(u\in X\vee v\in B)} \wedge \underbrace{(u\in X\vee v\in A)} \wedge (v\in A\cup B)$

$p\wedge q \Rightarrow p$
$p\wedge q \Leftarrow p$ X

$\Rightarrow u\in X \wedge v\in A\cup B \Leftrightarrow (u,v)\in X\times(A\cup B)$

овде је само импликација

$\boxed{\Leftarrow}$ $(u,v)\in X\times(A\cup B) \Leftrightarrow u\in X \wedge v\in A\cup B$

$\Leftrightarrow u\in X \wedge (v\in A \vee v\in B)$

$\Leftrightarrow (u\in X\wedge v\in A)\vee(u\in X\wedge v\in B)$

$\Leftrightarrow (u,v)\in X\times A \vee (u,v)\in X\times B$

$\Leftrightarrow (u,v)\in (X\times A)\cup(X\times B)$

могли смо само ово да смо видели одмах

• бинарна релација – одабир подскупа од $A\times B$ на $A\times B$

• њен график је $\Gamma\subseteq A\times B$, $\Gamma\in\mathcal{P}(A\times B)$

• инверзни график $\Gamma^{-1}=\{(b,a)\in B\times A \mid (a,b)\in\Gamma\}$

• композиција релација

$\Gamma_1\subseteq A\times B$ и $\Gamma_2\subseteq B\times C$

$$\Gamma_2\circ\Gamma_1=\{(a,c)\in A\times C \mid (\exists b\in B)(a,b)\in\Gamma_1 \wedge (b,c)\in\Gamma_2\}$$

$A=\{0,1\}$

$B=\{2,3\}$

B \ A	0	1
2	(0,2)	(1,2)
3	(0,3)	(1,3)

$\Gamma=\{(1,2),(0,3)\}$

$\Gamma^{-1}=\{(2,1),(3,0)\}$

5 Домаћи: Показати да важи

(1) $(\Gamma_2\circ\Gamma_1)^{-1}=\Gamma_1^{-1}\circ\Gamma_2^{-1}$

(2) $\Gamma_1\circ(\Gamma_2\circ\Gamma_3)=(\Gamma_1\circ\Gamma_2)\circ\Gamma_3$

(1) $(c,a)\in(\Gamma_2\circ\Gamma_1)^{-1} \Leftrightarrow (a,c)\in\Gamma_2\circ\Gamma_1$

$(a,b)\in\Gamma_1 \wedge (b,c)\in\Gamma_2$

(1) $(c,a) \in (\Gamma_2 \circ \Gamma_1)^{-1} \iff (a,c) \in \Gamma_2 \circ \Gamma_1$

$\iff (\exists b \in B)\ (a,b) \in \Gamma_1 \wedge (b,c) \in \Gamma_2$

$\iff (\exists b \in B)\ (b,a) \in \Gamma_1^{-1} \wedge (c,b) \in \Gamma_2^{-1}$

$\iff (\exists b \in B)\ (c,b) \in \Gamma_2^{-1} \wedge (b,a) \in \Gamma_1^{-1}$

$\iff (c,a) \in \Gamma_1^{-1} \circ \Gamma_2^{-1}$

(2) $\Gamma_3 \subseteq A \times B$ $\qquad \Gamma_1 \circ (\Gamma_2 \circ \Gamma_3) \subseteq A \times D$

$\Gamma_2 \subseteq B \times C$

$\Gamma_1 \subseteq C \times D$

$(a,d) \in \Gamma_1 \circ (\Gamma_2 \circ \Gamma_3) \iff (\exists c \in C)\ (a,c) \in \Gamma_2 \circ \Gamma_3 \wedge (c,d) \in \Gamma_1$

$\iff (\exists c \in C)(\exists b \in B)\ (a,b) \in \Gamma_3 \wedge (b,c) \in \Gamma_2 \wedge (c,d) \in \Gamma_1$

$\iff (\exists b \in B)(\exists c \in C)\ (b,c) \in \Gamma_2 \wedge (c,d) \in \Gamma_1 \wedge (a,b) \in \Gamma_3$

$\iff (\exists b \in B)\ (b,d) \in \Gamma_1 \circ \Gamma_2 \wedge (a,b) \in \Gamma_3$

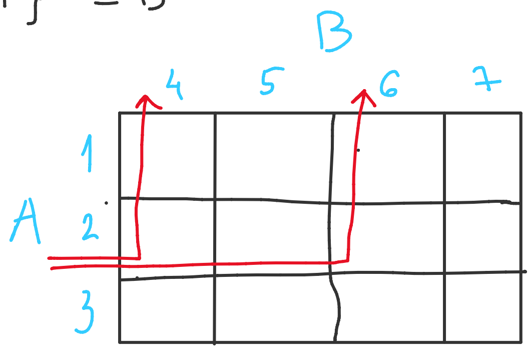
$\iff (a,d) \in (\Gamma_1 \circ \Gamma_2) \circ \Gamma_3$

• Засек графика $\Gamma \in \mathcal{P}(A \times B)$ дуж скупа $S \subseteq A$.

$$\Gamma(S) = \{ b \in B \mid (\exists a \in S)\ (a,b) \in \Gamma \} \subseteq B$$

Конкретан пример: $A = \{1,2,3\}$

$B = \{4,5,6,7\}$



$\Gamma = \{(1,4),(1,7),(2,4),(2,6)\} \subseteq A \times B$

$\Gamma^{-1} = \{(4,1),(7,1),(4,2),(6,2)\} \subseteq B \times A$

$\boxed{\Gamma(\{2\}) = \Gamma(2) = \{4,6\}}$ (jер $(2,4),(2,6) \in \Gamma$)

$\Gamma(\{2,3\}) = \Gamma(2) = \{4,6\}$

$\Gamma(\{3\}) = \Gamma(3) = \emptyset$

$\Gamma(\{1,2\}) = \{4,6,7\}$ $\qquad \Gamma(A) = \{4,6,7\} \neq B$

• Пројекција графика

• пројекције графика

$$\pi_1\Gamma = \{a \in A \mid (\exists b \in B)\ (a,b) \in \Gamma\}$$

$$\pi_2\Gamma = \{b \in B \mid (\exists a \in A)\ (a,b) \in \Gamma\}$$

важи: $\Gamma(A) = \pi_2\Gamma$

6

Домаћи: Показати следећа својства засека

(1) $\Gamma(A) = \Gamma(A \cap \pi_1\Gamma)$

(2) $(\Gamma_2 \circ \Gamma_1)(S) = \Gamma_2(\Gamma_1(S))$ за све $S \subseteq A$

(3) $\pi_1(\Gamma_2 \circ \Gamma_1) = \Gamma_1^{-1}(\pi_1\Gamma_2)$

(4) $\pi_2(\Gamma_2 \circ \Gamma_1) = \Gamma_2(\pi_2\Gamma_1)$

(5) За $\mathcal{A} \subset \mathcal{P}(A)$ важи $\Gamma\left(\bigcup_{X \in \mathcal{A}} X\right) = \bigcup_{X \in \mathcal{A}} \Gamma(X)$

(6) $\Gamma(\phi) = \phi$

(7) $S\cancel{A} \subseteq \cancel{B}T \Rightarrow \Gamma(\overset{S}{\cancel{A}}) \subseteq \Gamma(\overset{T}{\cancel{B}})$

(8) $\Gamma(\underset{S}{\cancel{A}} \cap \underset{T}{\cancel{B}}) \subseteq \Gamma(\underset{S}{\cancel{A}}) \cap \Gamma(\underset{T}{\cancel{B}})$

овде не важи увек једнакост