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Редови са позитивним члановима

$\sum_{n=1}^{\infty} a_n \quad , a_n \geqslant 0$ $(a_n \in \mathbb{R})$ $\qquad S_N = \sum_{n=1}^{N} a_n$ растући низ

$\circledast$ $\sum_{n=1}^{\infty} a_n$ конв акко је низ S_N ограничен

(пишемо $\sum_{n=1}^{\infty} a_n < +\infty$ кад конв. $\sum_{n=1}^{\infty} a_n = +\infty$)

Критеријуми:

$\boxed{1}$ Поредбени критеријум (ПК):

(а) $a_n \leq M b_n \quad , \forall n \geqslant n_0 \quad$, онда $\qquad \left.\begin{array}{l} \sum b_n < \infty \Rightarrow \sum a_n < \infty \\ \sum a_n = \infty \Rightarrow \sum b_n = \infty \end{array}\right\} (*)$

$M > 0$

(б) $a_n = O(b_n) \quad , n \to \infty \Rightarrow (*)$

(в) $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \gamma \in \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$

(1) $\gamma < \infty \Rightarrow (*)$

(2) $\gamma > 0 \Rightarrow \lim\limits_{n\to\infty} \frac{b_n}{a_n} = \frac{1}{\gamma} < \infty \Rightarrow (*)$ када a_n и b_n замене места

(3) $0 < \gamma < \infty \Rightarrow$ важи „$\Leftrightarrow$" уместо „$\Rightarrow$" у $(*)$.

$\big((3) \Leftrightarrow (1) \wedge (2)\big)$

(г) $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n} \Rightarrow (*)$

$\forall n \geqslant n_0$

$\boxed{2}$ Даламберов и Кошијев критеријум

(а) Даламбер: (1) $\frac{a_{n+1}}{a_n} \leq q < 1 \quad , \forall n \geqslant n_0 \Rightarrow \sum a_n < \infty$

$\frac{a_{n+1}}{a_n} \geqslant 1 \quad , \forall n \geqslant n_0 \Rightarrow \sum a_n = \infty$

(2) $\exists \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = q$ $\qquad$ (можемо и $\overline{\lim}$ уместо $\lim$ да гледамо, али само за $q < 1$ кад конв.)

$q < 1 \Rightarrow \sum a_n < \infty$

$q > 1 \Rightarrow \sum a_n = \infty$

$q = 1$ не знамо!

(б) Коши,

уместо $\frac{a_{n+1}}{a_n}$ посматрамо $\sqrt[n]{a_n}$ и све је исто

Напомена: $\underbrace{\underline{\lim}\, \frac{a_{n+1}}{a_n}} \leq \underbrace{\underline{\lim}\, \sqrt[n]{a_n}} \leq \underbrace{\overline{\lim}\, \sqrt[n]{a_n}} \leq \underbrace{\overline{\lim}\, \frac{a_{n+1}}{a_n}}$ Даламбер Коши

Ако $\exists \lim$ код Даламбера $\underline{\lim} = \overline{\lim} \Rightarrow \underline{\lim} = \overline{\lim} \Rightarrow \lim \frac{a_{n+1}}{a_n} = \lim \sqrt[n]{a_n}$

Даламбер $\Rightarrow$ Коши

обрнуто не важи: $a_n = \begin{cases} \frac{1}{2^n}, & 2 \mid n \\ \frac{1}{3^n}, & 2 \nmid n \end{cases}$

$\frac{a_{n+1}}{a_n} = \begin{cases} \frac{2^n}{3^{n+1}}, & 2 \mid n \\ \frac{3^n}{2^{n+1}}, & 2 \nmid n \end{cases}, \quad \sqrt[n]{a_n} = \begin{cases} \frac{1}{2}, & 2 \mid n \\ \frac{1}{3}, & 2 \nmid n \end{cases}$

$\overline{\lim}\, \frac{a_{n+1}}{a_n} = \infty$ Даламбер не ради

$\overline{\lim}\, \sqrt[n]{a_n} = \frac{1}{2}$ Коши ради

$\Rightarrow \sum a_n < \infty$

① Доказати $\sum \frac{1}{n^2} < \infty$.

прошли час: $\underline{\underline{\sum \frac{1}{n^2-1} < \infty}}$.

Пошто $\lim\limits_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^2-1}} = \lim\limits_{n\to\infty} \frac{n^2-1}{n^2} = 1 \overset{\text{ПК (в)}}{\Longrightarrow} \sum \frac{1}{n^2} < \infty$

Не важи $\sum \frac{1}{n^2-1} = \sum \frac{1}{n^2}$!

Анализа 2, $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$.

② $\sum \frac{n^5}{2^n+3^n}$ - испитати конвергенцију реда.

$\frac{n^5}{2^n+3^n} \to 0$

$\frac{n^5}{2^n+3^n} \overset{?}{\leq} \frac{1}{n^2} \Leftrightarrow 2^n+3^n \geq n^7$

$\frac{3^n}{n^7} \xrightarrow[n\to\infty]{} \infty \Rightarrow \frac{3^n}{n^7} \geq 1, \forall n \geq n_0$

$\left(\frac{a^n}{n^k} \to \infty\right)$

$\backslash\, n^k \quad ,$

$2^n+3^n \geq 3^n \geq n^7$, $\forall n \geq n_0$

$\frac{n^5}{2^n+3^n} \le \frac{1}{n^2}$ и $\sum \frac{1}{n^2}<\infty \overset{\text{ПК(a)}}{\Longrightarrow} \sum \frac{n^5}{2^n+3^n}<\infty$.

③ $\sum\limits_{n=2}^{\infty}\left(\frac{n-1}{n+1}\right)^{n(n-1)}$

$$\sqrt[n]{a_n} = \sqrt[n]{\left(\frac{n-1}{n+1}\right)^{n(n-1)}} = \left(\frac{n-1}{n+1}\right)^{n-1} = \left(1+\frac{-2}{n+1}\right)^{(n-1)\cdot\frac{n+1}{-2}\cdot\frac{-2}{n+1}} = \Bigg(\underbrace{\left(1-\frac{2}{n+1}\right)^{-\frac{n+1}{2}}}_{\to e}\Bigg)^{-2\overset{\to 1}{\frac{n-1}{n+1}}} \xrightarrow[n\to\infty]{} e^{-2} = \frac{1}{e^2}<1$$

$\overset{\text{Коши}}{\Longrightarrow}$ конв.

④ $\sum \underbrace{n^3\cdot\sin\frac{\pi}{3^n}}_{a_n}$

$0<\frac{\pi}{3^n}\le\frac{\pi}{3} \Rightarrow \sin\left(\frac{\pi}{3^n}\right)\in\left(0,\sin\frac{\pi}{3}\right] = \left(0,\frac{\sqrt{3}}{2}\right] \Rightarrow n^3\cdot\sin\frac{\pi}{3^n}>0$

$\boxed{\lim\limits_{x\to0}\frac{\sin x}{x}=1, \quad \sin x\sim x, x\to0}$

$b_n = n^3\cdot\frac{\pi}{3^n}$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim\frac{\cancel{n^3}\cdot\sin\left(\frac{\pi}{3^n}\right)}{\cancel{n^3}\cdot\underbrace{\frac{\pi}{3^n}}_{\to0}} = 1$$

$\Rightarrow \sum a_n<\infty \overset{\text{ПК(В)}}{\Longleftrightarrow} \sum b_n<\infty$ (еквиконвергентни)

$\mu=1$: $a_n\sim b_n, n\to\infty$

ПК(В)

$\lim\frac{a_n}{b_n}=1 \Rightarrow a_n$ и b_n су еквиконвергентни

$\sum b_n = \sum \pi\cdot n^3\cdot\frac{1}{3^n}$

$$\sqrt[n]{b_n} = \underbrace{\sqrt[n]{\pi}}_{\downarrow\, 1}\cdot\underbrace{\Big(\underbrace{\sqrt[n]{n}}_{\downarrow\, 1}\Big)^3}_{\downarrow\, 1}\cdot\frac{1}{3} \xrightarrow[n\to\infty]{} \frac{1}{3}<1 \overset{\text{Коши}}{\Rightarrow} \sum b_n<\infty \Rightarrow \sum a_n<\infty.$$

$\boxed{\text{Т}}$ $\sum\limits_{n=1}^{\infty} \frac{1}{n^p} < \infty \iff p > 1$

$p \in \mathbb{R}$

$\sum \frac{1}{n^2} < \infty \quad , 2 > 1$

$\sum \frac{1}{\sqrt{n}} = \infty \quad , \frac{1}{2} \leq 1$

$\sum \frac{1}{n} = \infty \quad , 1 \leq 1$

→ хармонијски ред

⑤ $\sum \frac{1}{\sqrt{n(n+1)}}$

$$\frac{1}{\sqrt{n(n+1)}} \sim \frac{1}{n} \text{, јер } \frac{\frac{1}{\sqrt{n(n+1)}}}{\frac{1}{n}} = \frac{n}{\sqrt{n(n+1)}} = \sqrt{\frac{n}{n+1}} \to 1$$

$$\sum \frac{1}{n} = \infty \Rightarrow \sum \frac{1}{\sqrt{n(n+1)}} = \infty$$

⑥ $\sum \frac{1}{n^n}$

$$\left.\begin{array}{l} n^n \geq n^2 \;, n \geq 2 \\ 1^1 \geq 1^2 \end{array}\right\} \quad \frac{1}{n^n} \leq \frac{1}{n^2} \quad \xRightarrow{\text{ПК}} \sum \frac{1}{n^n} < \infty$$

Домаћи: а) $\sum \frac{n^{n+\frac{1}{n}}}{\left(n+\frac{1}{n}\right)^n}$ б) $\sum \frac{1}{\sqrt[n]{\log n}}$ в) $\sum \frac{2^n}{2^n+n}$

⑦ $\sum\limits_{n=2}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right)^p \cdot \log \frac{n-1}{n+1}$, у зависности од $p \in \mathbb{R}$

$\left(\sqrt{n+1} - \sqrt{n}\right)^p \geq 0$

$\frac{n-1}{n+1} < 1 \Rightarrow \log \frac{n-1}{n+1} < 0$

ред са нег. члановима →помножимо са -1 ⇒ ред. са поз. члановима

$$a_n = \left(\sqrt{n+1} - \sqrt{n}\right)^p \cdot \underline{\log\left(1 - \frac{2}{n+1}\right)} = \left(\sqrt{n}\left(\sqrt{\frac{n+1}{n}} - 1\right)\right)^p \cdot \underline{\left(-\frac{2}{n+1} + \overset{\overset{o\left(\frac{1}{n}\right)}{\shortparallel}}{o\left(-\frac{2}{n+1}\right)}\right)} =$$

$$= n^{p/2} \cdot \left(\left(\underline{1+\frac{1}{n}}\right)^{1/2} - 1\right)^p \cdot \left(-\frac{2}{n+1} + o\left(\frac{1}{n}\right)\right) =$$

$$= n^{p/2} \cdot \left(\underline{1 + \frac{1}{2} \cdot \frac{1}{n}} - 1 + \underline{o\left(\frac{1}{n}\right)}\right)^p \cdot \left(-\frac{2}{n+1} + o\left(\frac{1}{n}\right)\right) =$$

$$= n^{p/2}\cdot\left(\frac{1}{2n}+o\left(\frac{1}{n}\right)\right)^{p}\cdot\left(-\frac{2}{n+1}+o\left(\frac{1}{n}\right)\right)=$$

$$= n^{p/2}\cdot\left(\frac{1}{2n}\right)^{p}\cdot\underbrace{\boxed{\left(1+o(1)\right)^{p}}}_{\sim 1}\cdot\left(-\frac{2}{n+1}+o\left(\frac{1}{n}\right)\right)\sim$$

$$\sim\frac{1}{2^{p}\cdot n^{p/2}}\cdot\underbrace{\left(-\frac{2}{n+1}\right)}_{\sim -\frac{2}{n}}\sim-\frac{1}{2^{p-1}\cdot n^{p/2+1}}\quad ,n\to\infty$$

a_n је еквиконв. са $\frac{1}{n^{p/2+1}}$

$$\sum\frac{1}{n^{p/2+1}}<\infty\iff\frac{p}{2}+1>1\iff\frac{p}{2}>0\iff p>0$$

⑧ $\sum\frac{1}{n^{1+\frac{1}{n}}}$

$1+\frac{1}{n}>1$, али није константа, не можемо да применимо $\boxed{T}$

(Т би рекла да конв!)

$$\frac{1}{n^{1+\frac{1}{n}}}=\frac{1}{n\cdot n^{\frac{1}{n}}}=\frac{1}{n\cdot\underbrace{\sqrt[n]{n}}_{\sim 1}}\sim\frac{1}{n}\quad ,n\to\infty$$

$$\sum\frac{1}{n}=\infty\implies\sum\frac{1}{n^{1+\frac{1}{n}}}=\infty$$

$\boxed{T}$ $\sum\frac{1}{e^{rn}\cdot n^{p}\cdot(\log n)^{q}}<\infty\iff$

$p,q,r\in\mathbb{R}$

1° $r>0$

2° $r=0\ \wedge\ p>1$

3° $r=0\ \wedge\ p=1\ \wedge\ q>1$

спец: $r=0, q=0$ $\frac{1}{n^{p}}$, $p>1$

пр. $\sum\frac{1}{n(\log n)^{2}}<\infty$, $r=0$, $p=1$, $q=2$

$\sum\frac{1}{n\cdot\log n}=\infty$, $r=0$, $p=1$, $q=1$

⑨ $$\sum_{n=2}^{\infty} \frac{1-\cos\frac{1}{\sqrt{n}}}{(\log n)^p}$$

$1-\cos\frac{1}{\sqrt{n}} \geqslant 0$ → поз. чланове

$$\cos\frac{1}{\sqrt{n}} = 1-\frac{1}{2}\left(\frac{1}{\sqrt{n}}\right)^2+\frac{1}{24}\left(\frac{1}{\sqrt{n}}\right)^4+o\left(\frac{1}{n^2}\right) = 1-\frac{1}{2n}+\cancel{\frac{1}{24n^2}}+o\left(\frac{1}{n^2}\right) = 1-\frac{1}{2n}+o\left(\frac{1}{n}\right),\ n\to\infty$$

$$\frac{1-\cos\frac{1}{\sqrt{n}}}{(\log n)^p} = \frac{1-\left(1-\frac{1}{2n}+o\left(\frac{1}{n}\right)\right)}{(\log n)^p} = \frac{\frac{1}{2n}+o\left(\frac{1}{n}\right)}{(\log n)^p} \sim \frac{1}{2n(\log n)^p},\ n\to\infty$$

Почетни конв. $\Leftrightarrow \frac{1}{2n(\log n)^p} \Leftrightarrow p>1$

⑩ $$\sum \underbrace{(\underbrace{\sqrt[n]{n}-1}_{>0})^{\alpha}\cdot(\underbrace{\sin\tfrac{1}{n}}_{>0})^{\beta}}_{a_n}$$

$$\sqrt[n]{n} = e^{\frac{1}{n}\log n} = 1+\frac{\log n}{n}+o\left(\frac{\log n}{n}\right)$$

$$\sin\frac{1}{n} = \frac{1}{n}+o\left(\frac{1}{n}\right)$$

$$\left(\sqrt[n]{n}-1\right)^{\alpha} \sim \left(\frac{\log n}{n}\right)^{\alpha}$$

$$\left(\sin\frac{1}{n}\right)^{\beta} \sim \left(\frac{1}{n}\right)^{\beta}$$

$$a_n \sim \left(\frac{\log n}{n}\right)^{\alpha}\cdot\left(\frac{1}{n}\right)^{\beta} = \frac{1}{n^{\alpha+\beta}\cdot(\log n)^{-\alpha}}$$

$$\sum a_n<\infty \Leftrightarrow (\alpha+\beta>1) \vee \left(\alpha+\beta=1 \wedge \underbrace{-\alpha>1}_{\alpha<-1}\right)$$

Домаћи:

① $\sum \frac{\operatorname{arctg} n}{n^2+1}$

② $\sum \frac{1}{\sqrt{n}\cdot 2^{\sqrt{n}}}$

③ $\sum a_n$ у зависности од $p\in\mathbb{R}$, $a_n = \frac{n+1}{\sqrt{n^2+n+1}+n} - \frac{1}{2} - \frac{p}{n}$