спец.: $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1.$

(1) $$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\log x \cdot \frac{1}{x}} = e^{0} = 1.$$

? $\lim\limits_{x\to\infty} \dfrac{\log x}{x}$

$$\left.\begin{array}{l}\lim\limits_{x\to\infty} \dfrac{(\log x)'}{(x)'} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}}{1} = \lim\limits_{x\to\infty} \dfrac{1}{x} = 0 \\ \lim\limits_{x\to\infty} x = \infty\end{array}\right\} \xRightarrow{\text{ЛП}} \lim_{x\to\infty} \frac{\log x}{x} = 0$$

(2) а) $\lim\limits_{x\to 0} \dfrac{x^2 \sin\frac{1}{x}}{\sin x}$

ЛП: $$\lim_{x\to 0} \frac{\left(x^2 \sin\frac{1}{x}\right)'}{(\sin x)'} = \lim_{x\to 0} \frac{x^2 \cdot \cos\frac{1}{x} \cdot \left(-\frac{1}{x^2}\right) + 2x \cdot \sin\frac{1}{x}}{\cos x} = \lim_{x\to 0} \frac{\overset{\to 0}{2x \cdot \sin\frac{1}{x}} - \overset{?}{\cos\frac{1}{x}}}{\underset{\to 1}{\cos x}}$$ X

зад. једно решење:

$$\lim_{x\to 0} \frac{x^2 \sin\frac{1}{x}}{\sin x} = \lim_{x\to 0} \frac{\overset{\to 0}{x \sin\frac{1}{x}}}{\underset{\to 1}{\frac{\sin x}{x}}} = \frac{0}{1} = 0.$$

б) $\lim\limits_{x\to\infty} \dfrac{x - \sin x}{x + \sin x}$

ЛП: $$\lim_{x\to\infty} \frac{(x - \sin x)'}{(x + \sin x)'} = \lim_{x\to\infty} \frac{1 - \cos x}{1 + \cos x} \quad \nexists$$

$$\lim_{x\to\infty} \frac{1 - \cancel{\frac{\sin x}{x}}^{\to 0}}{1 + \cancel{\frac{\sin x}{x}}_{\to 0}} = 1.$$

$(x^x)' = (e^{x \cdot \log x})' =$

$= e^{x \log x} \cdot \left(x \cdot \frac{1}{x} + 1 \cdot \log x\right) =$

$= x^x \cdot (1 + \log x)$

(3) $$\lim_{x\to 1} \frac{x^x - x}{\log x - x + 1} = -2$$

ЛП: $\lim\limits_{x\to 1} \log x - x + 1 = 0$

$\lim\limits_{x\to 1} x^x - x = 0$

$$\lim_{x\to 1} \frac{(x^x - x)'}{(\log x - x + 1)'} = \lim_{x\to 1} \frac{x^x(1 + \log x) - 1}{\frac{1}{x} - 1} = ? = -2$$

ЛП: $\lim\limits_{x\to 1} x^x(1 + \log x) - 1 = 0$

$$\overset{\to 1}{x^x \cdot \frac{1}{x}} + \overset{\to 1}{x^x}\overset{\to 1}{(1 + \log x)^2} \qquad 1 + 1$$

ЛП: $\lim\limits_{x\to1} x^x(1+\log x)-1=0$

$\lim\limits_{x\to1} \frac{1}{x}-1=0$

$$\lim_{x\to1}\frac{\overbrace{x^x\cdot\frac{1}{x}}^{\to1}+\overbrace{x^x}^{\to1}\overbrace{(1+\log x)^2}^{\to1}}{\underbrace{-\frac{1}{x^2}}_{\to-1}}=\frac{1+1}{-1}=-2.$$

(4) $$\lim_{x\to\infty}\frac{x^4+x^2-3}{e^x}=0$$

Важи и за произв. полином $P(x)$: $\lim\limits_{x\to\infty}\frac{P(x)}{e^x}=0.$

ЛП

$\lim\limits_{x\to\infty} e^x=\infty$ $\quad\lim\limits_{x\to\infty}\frac{4x^3-2x}{e^x}=0$

ЛП

$\lim\limits_{x\to\infty} e^x=\infty$ $\quad\lim\limits_{x\to\infty}\frac{12x^2-2}{e^x}=0$

ЛП

$\lim\limits_{x\to\infty} e^x=\infty$ $\quad\lim\limits_{x\to\infty}\frac{24x}{e^x}=0$

ЛП

$\lim\limits_{x\to\infty} e^x=\infty$ $\quad\lim\limits_{x\to\infty}\frac{\overbrace{24}^{\to24}}{\underbrace{e^x}_{\to\infty}}=0$

домаћи: ⊛ $\lim\limits_{x\to0}\left(\operatorname{ctg}x-\frac{1}{x}\right)$

⊛ $\lim\limits_{x\to\infty}\frac{x}{q^x}$, $q>1$

⊛ $\lim\limits_{x\to\infty}\frac{x^a}{q^x}$, $a,q>1$

↳ помоћ: $\frac{x^a}{q^x}=\left(\frac{x}{q^{\frac{x}{a}}}\right)^a$

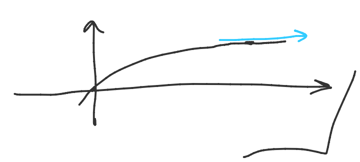
(5) $f:(0,+\infty)\to\mathbb{R}$ је диф. ф-ја и $\lim\limits_{x\to\infty}(f(x)+f'(x))=A$. Доказати $\lim\limits_{x\to\infty} f(x)=A$.

$$(f(x)+f'(x))\cdot e^x=e^xf(x)+e^xf'(x)=(e^x)'\cdot f(x)+e^x\cdot f'(x)=\underline{(e^x\cdot f(x))'}$$

$F(x)=e^x\cdot f(x)$

$G(x)=e^x$

Следи ли $\lim\limits_{x\to\infty}f'(x)=0$



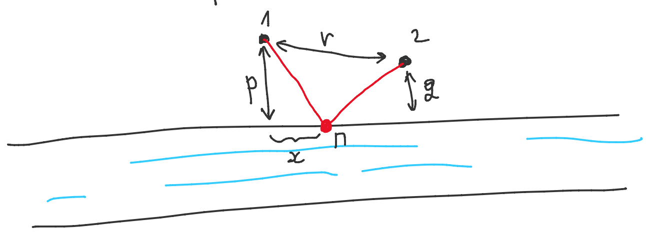
$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{F(x)}{G(x)}=A.$$

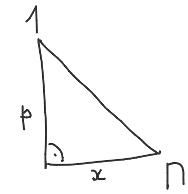
ЛП

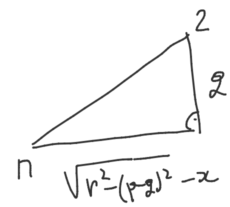
$\lim\limits_{x\to\infty}G(x)=\infty$

$$\lim_{x\to\infty}\frac{F'(x)}{G'(x)}=\frac{(e^x\cdot f(x))'}{(e^x)'}=\lim_{x\to\infty}\frac{\cancel{e^x}(f(x)+f'(x))}{\cancel{e^x}}=\lim_{x\to\infty}(f(x)+f'(x))=A.$$

⑥ Становници два града која се налазе на истој страни обале реке која тече право граде пумпу. Растојање између градова је r, првог гр. и реке је p, другог гр. и реке је q. Доказати да је дужина цеви бар $\sqrt{r^2+4pq}$.

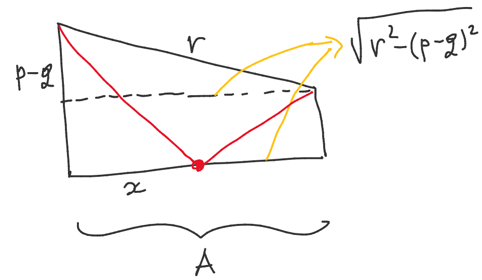






$$f(x)=d(1,П)+d(П,2)=\sqrt{x^2+p^2}+\sqrt{\left(\sqrt{r^2-(p-q)^2}-x\right)^2+q^2}$$

$$\left(\sqrt{\left(\sqrt{r^2-(p-q)^2}-x\right)^2+q^2}=\sqrt{r^2-(p-q)^2\underbrace{-2\sqrt{r^2-(p-q)^2}}\cdot x+x^2+q^2}\right)$$



$$f'(x)=\frac{1}{\cancel{2}\sqrt{x^2+p^2}}\cdot\cancel{2}x+\frac{1}{\cancel{2}\sqrt{\left(\sqrt{r^2-(p-q)^2}-x\right)^2+q^2}}\cdot\left(\cancel{2}x-\cancel{2}\sqrt{r^2-(p-q)^2}\right)=$$

$$=\frac{x}{\sqrt{x^2+p^2}}+\frac{x-\sqrt{r^2-(p-q)^2}}{\sqrt{\,---}}$$

$$\sqrt{r^2-(p-q)^2}=A$$

$f'(x)=0$

$$\frac{x}{\sqrt{x^2+p^2}}+\frac{x-A}{\sqrt{(A-x)^2+q^2}}=0\Rightarrow\frac{x^2}{x^2+p^2}=\frac{(x-A)^2}{(x-A)^2+q^2}$$

$$\Rightarrow \cancel{x^2(x-A)^2}+x^2q^2=\cancel{x^2(x-A)^2}+p^2(x-A)^2$$

$$\left(\frac{x}{x-A}\right)^2=\left(\frac{p}{q}\right)^2$$

$x\in(0,A)$: $\quad\frac{x}{A-x}=\frac{p}{q}\Rightarrow xq=Ap-xp$

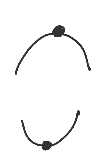
(због логике задатка)

$$x(p+q)=Ap\Rightarrow x=\frac{Ap}{p+q}$$

Ако је f два пута диф. и $f'(x_0)=0$, $f''(x_0)\neq 0$

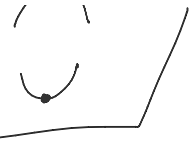
1) $f''(x_0)<0\Rightarrow$ лок. макс.

2) $f''(x_0)>0\Rightarrow$ лок. мин.



1) $f''(x_0) < 0 \Rightarrow$ лок. макс.

2) $f''(x_0) > 0 \Rightarrow$ лок. мин.



$$f''(x) = \frac{\sqrt{x^2+p^2} - x\cdot\frac{1}{\cancel{2}\sqrt{x^2+p^2}}\cdot\cancel{2}x}{x^2+p^2} + \frac{\sqrt{(x-A)^2+q^2} - \frac{\cancel{2}(x-A)}{\cancel{2}\sqrt{(x-A)^2+q^2}}\cdot(x-A)}{(x-A)^2+q^2} =$$

$$= \frac{p^2}{(x^2+p^2)^{3/2}} + \frac{q^2}{\left((x-A)^2+q^2\right)^{3/2}}$$

$$f''\left(\frac{pA}{p+q}\right) = \frac{p^2}{(p^2)^{3/2}\left(\frac{A}{p+q}+1\right)^{3/2}} + \frac{q^2}{\left(\left(\frac{pA}{p+q}-A\right)^2+q^2\right)^{3/2}} > 0 \Rightarrow \frac{pA}{p+q} \text{ је лок. мин.}$$

$$f(x) \geq f\left(\frac{pA}{p+q}\right) = \sqrt{\frac{p^2A^2}{(p+q)^2}+p^2} + \sqrt{\left(\frac{pA}{p+q}-A\right)^2+q^2} =$$

$$= \frac{p}{p+q}\sqrt{A^2+(p+q)^2} + \sqrt{A^2\frac{q^2}{(p+q)^2}+q^2} =$$

$$= \underbrace{\frac{p}{p+q}}\sqrt{r^2-(p-q)^2+(p+q)^2} + \underbrace{\frac{q}{p+q}}\sqrt{r^2-(p-q)^2+(p+q^2)} =$$

$$= \sqrt{r^2-(2p)\cdot(2q)} = \sqrt{r^2-4pq}$$

(*) Из Лагранжа: $f : \underbrace{I}_{\text{интервал}} \to \mathbb{R}$ диф. и $f'(x)=0,\ \forall x\in I \Rightarrow f = \text{const}$.

$$f(x)-f(y) = \underbrace{f'(c)}_{0}\cdot(x-y) = 0 \Rightarrow f(x) = f(y).$$

(7) Доказати:

а) $\operatorname{arctg} x + \operatorname{arctg}\frac{1}{x} = \frac{\pi}{2}$, $\forall x \in (0,+\infty)$

домаћи б) $2\operatorname{arctg} x + \arcsin\frac{2x}{1+x^2} = \pi\cdot\operatorname{sgn} x$, $|x|\geq 1$.

a) $f(x) = \operatorname{arctg} x + \operatorname{arctg}\frac{1}{x}$, $f : \mathbb{R}\setminus\{0\} \to \mathbb{R}$

f диф.

$$f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1}{x}\right)^2}\cdot\left(-\frac{1}{x^2}\right) = \frac{1}{1+x^2} - \frac{1}{x^2+1} = 0$$

$f = const$ НА ИНТЕРВАЛУ! $\mathbb{R}\setminus\{0\} = (-\infty,0)\cup(0,\infty)$

$(0,+\infty)$: $f(x) = c_1$

$$c_1 = f(1) = \underbrace{\operatorname{arctg} 1}_{\frac{\pi}{4}} + \operatorname{arctg} 1 = \frac{\pi}{2} \Rightarrow \operatorname{arctg} x + \operatorname{arctg}\frac{1}{x} = \frac{\pi}{2},\ x>0$$

или:

$$\left(\lim_{x\to\infty} f(x) = \frac{\pi}{2} + 0 = \frac{\pi}{2}\ ;\ \lim_{x\to 0+} f(x) = 0 + \frac{\pi}{2} = \frac{\pi}{2}\right)$$

$(-\infty,0)$: $f(x) = c_2$ (не мора $c_1 = c_2$)

$$c_2 = \lim_{x\to-\infty} f(x) = -\frac{\pi}{2} + 0 = -\frac{\pi}{2}$$

$$\Rightarrow \operatorname{arctg} x + \operatorname{arctg}\frac{1}{x} = -\frac{\pi}{2},\ x<0$$

⑧ $f(x) = \operatorname{arctg} x$

Наћи $f^{(n)}(0)$.

$$(e^x)^{(n)} = e^x$$

$$(\sin x)^{(4n)} = \sin x$$

$$(\sin x)^{(4n+2)} = -\sin x$$

$$f'(x) = \frac{1}{1+x^2}$$

$$f''(x) = -\frac{1}{(1+x^2)^2}\cdot 2x = -\frac{2x}{(1+x^2)^2}$$

$$(1+x^2)f''(x) + 2x\cdot f'(x) = 0 \quad /^{(n-2)}$$

$$\left((1+x^2)f''(x)\right)^{(n-2)} + \left(2x\cdot f'(x)\right)^{(n-2)} = 0$$

$$\left((1+x^2)\, f''(x)\right)^{(n-2)} + \left(2x\cdot f'(x)\right)^{(n-2)} = 0$$

$$\sum_{k=0}^{n-2}\binom{n-2}{k}(1+x^2)^{(k)}\cdot\left(f''\right)^{(n-2-k)}(x) + \sum_{k=0}^{n-2}\binom{n-2}{k}(2x)^{(k)}\left(f'\right)^{(n-2-k)}(x) = 0$$

$(1+x^2)' = 2x$

$(1+x^2)'' = 2$

$(1+x^2)^{(k)} = 0,\ k\geq 3$

$(2x)' = 2$

$(2x)^{(k)} = 0\ ,\ k\geq 2$

$$\underbrace{\binom{n-2}{0}(1+x^2)\, f^{(n)}(x)}_{k=0} + \underbrace{\binom{n-2}{1}\cdot \overset{0}{\overset{\shortparallel}{2x}}\cdot f^{(n-1)}(x)}_{k=1} + \underbrace{\binom{n-2}{2}\, 2\cdot f^{(n-2)}(x)}_{k=2} +$$

$$+\underbrace{\binom{n-2}{0}\cdot \underset{\underset{0}{\shortparallel}}{2x}\cdot f^{(n-1)}(x)}_{k=0} + \underbrace{\binom{n-2}{1}\cdot 2\cdot f^{(n-2)}(x)}_{k=1} = 0$$

$x=0$:

$$1\cdot 1\cdot f^{(n)}(0) + \frac{(n-2)(n-3)}{\cancel{2}}\cdot\cancel{2}\cdot f^{(n-2)}(0) + (n-2)\cdot 2\cdot f^{(n-2)}(0) = 0$$

$$f^{(n)}(0) + (n-2)(n-1)\, f^{(n-2)}(0) = 0 \Rightarrow f^{(n)}(0) = -(n-2)(n-1)\, f^{(n-2)}(0).$$

$$f(0) = f^{(0)}(0) = \operatorname{arctg} 0 = 0 \Rightarrow f^{(2)} = 0 \Rightarrow f^{(4)}(0) = 0 \Rightarrow \dots\ f^{(2k)}(0) = 0\ ,\ \forall k\in\mathbb{N}_0$$

$$f'(0) = \frac{1}{1+0^2} = 1 \Rightarrow f^{(3)}(0) = -1\cdot 2\cdot f'(0) = -2$$

$$f^{(2k+1)}(0) = -(2k-1)\cdot(2k)\ f^{(2k-1)}(0) =$$

$$= +\underline{(2k-1)}\,\underline{(2k)}\,\underline{(2k-3)}\,\underline{(2k-2)}\ f^{(2k-3)}(0) =$$

$$= \dots = (-1)^k\cdot(2k)!\ f'(0) = (-1)^k\cdot(2k)!$$