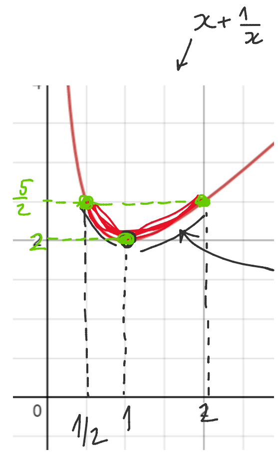
(1) $\int_{1/2}^{2}\left(1+x-\frac{1}{x}\right)\cdot e^{x+\frac{1}{x}}\,dx$



није бијекција
не можемо $x=\ldots$!

Хоћемо смену $t=x+\frac{1}{x}$, $x=\ldots?$ ↙ $\varphi(t)$

$g(x)=x+\frac{1}{x}$

$g'(x)=1-\frac{1}{x^2}=\frac{x^2-1}{x^2}>0 \iff x>1$

$[1/2,1]$ − g опада
$[1,2]$ − g расте
g бијекција на свакоме посебно

$$I=\int_{1/2}^{1}\left(1+x-\frac{1}{x}\right)\cdot e^{x+\frac{1}{x}}\,dx+\int_{1}^{2}\left(1+x-\frac{1}{x}\right)\cdot e^{x+\frac{1}{x}}\,dx$$

g^{-1}? $\quad x+\frac{1}{x}=t\ /\cdot x \Rightarrow x^2-xt+1=0$

$x=\_\_\quad x_{1/2}=\frac{t\pm\sqrt{t^2-4}}{2}$

опада

$$x=g^{-1}(t)=\frac{t-\sqrt{t^2-4}}{2},\quad x\in[1/2,1],\ t\in[2,5/2]$$

$$x=g^{-1}(t)=\frac{t+\sqrt{t^2-4}}{2},\quad x\in[1,2],\ t\in[2,5/2]$$

расте

$$I_1=\int_{1/2}^{1}\left(1+x-\frac{1}{x}\right)\cdot e^{x+\frac{1}{x}}\,dx=\left|\begin{array}{l} x=\frac{t-\sqrt{t^2-4}}{2},\quad x+\frac{1}{x}=t \\ dx=\frac{1-\frac{t}{\sqrt{t^2-4}}}{2}\,dt,\quad \begin{array}{c|c|c} x & 1/2 & 1 \\ \hline t & 5/2 & 2 \end{array}\end{array}\right|=$$

$$=\int_{5/2}^{2}\left(1+\frac{t-\sqrt{t^2-4}}{2}-\frac{2}{\cancel{t-\sqrt{t^2-4}}}\right)\cdot e^{t}\cdot\frac{\overset{-1}{\cancel{\sqrt{t^2-4}-t}}}{2\sqrt{t^2-4}}\,dt=$$

$$=\int_{5/2}^{2}(-1)\left(t-\sqrt{t^2-4}+\left(t-\sqrt{t^2-4}\right)^2\cdot\frac{1}{2}-2\right)\cdot\frac{e^t}{2\sqrt{t^2-4}}\,dt=$$

$$=\int^{5/2}\left(t-\sqrt{t^2-4}+\left(t^2+t^2-4\right)-t\sqrt{t^2-4}-2\right)\cdot\frac{e^t\,dt}{2\sqrt{t^2-4}}=$$

$$= \int_2^{5/2} \left(t - \sqrt{t^2-4} + \underbrace{(t^2+t^2-4}_{2}) - t\sqrt{t^2-4} - 2 \right) \cdot \frac{e^t\,dt}{2\sqrt{t^2-4}} =$$

$$\boxed{= \int_2^{5/2} \left(t - \sqrt{t^2-4} + t^2 - 4 - t\sqrt{t^2-4} \right) \cdot \frac{e^t\,dt}{2\sqrt{t^2-4}}}$$

$$I_2 = \int_1^2 \left(1 + x - \frac{1}{x}\right) \cdot e^{x+\frac{1}{x}}\,dx = \left/ \begin{array}{ll} x = \dfrac{t+\sqrt{t^2-4}}{2} & , \; x + \frac{1}{x} = t \\ dx = \dfrac{1 + \frac{t}{\sqrt{t^2-4}}}{2} & , \; \begin{array}{c|c|c} x & 1 & 2 \\ \hline t & 2 & 5/2 \end{array} \end{array} \right/ =$$

$$= \int_2^{5/2} \left(1 + \frac{t+\sqrt{t^2-4}}{2} - \frac{2}{t+\sqrt{t^2-4}} \right) \cdot e^t \cdot \frac{\sqrt{t^2-4}+t}{2\sqrt{t^2-4}}\,dt =$$

$$= \int_2^{5/2} \left(t + \sqrt{t^2-4} + \frac{1}{2}\left(t+\sqrt{t^2-4}\right)^2 - 2 \right) \cdot \frac{e^t\,dt}{2\sqrt{t^2-4}} =$$

$$\boxed{= \int_2^{5/2} \left(t + \sqrt{t^2-4} + t^2 - 4 + t\sqrt{t^2-4} \right) \cdot \frac{e^t\,dt}{2\sqrt{t^2-4}}}$$

$$I = I_1 + I_2 = \int_2^{5/2} \left(t + t^2 - 4 \right) \cdot \frac{e^t\,dt}{\sqrt{t^2-4}} = \int_2^{5/2} e^t \sqrt{t^2-4}\,dt + \boxed{\int_2^{5/2} \frac{t e^t\,dt}{\sqrt{t^2-4}}}$$

$$\int_2^{5/2} e^t \sqrt{t^2-4}\,dt = \left/ \begin{array}{ll} u = \sqrt{t^2-4} & \rightarrow du = \dfrac{t}{\sqrt{t^2-4}} \\ dv = e^t dt & \rightarrow v = e^t \end{array} \right/ =$$

$$= \sqrt{t^2-4} \cdot e^t \Big|_2^{5/2} - \boxed{\int_2^{5/2} e^t \cdot \frac{t}{\sqrt{t^2-4}}\,dt}$$

$$\Rightarrow I = \int + \int = \sqrt{t^2-4} \cdot e^t \Big|_2^{5/2} = \boxed{\frac{3}{2} \cdot e^{5/2}}$$

(2) $$\int_{-1}^{1} \frac{dx}{(1+x^2)\cdot(1+e^x)} = \int_{-1}^{0} + \int_0^1 = \int_0^1 \left[\frac{e^x}{(1+x^2)(1+e^x)} + \frac{1}{(1+x^2)(1+e^x)} \right] dx = \int_0^1 \frac{(e^x+1)\,dx}{(1+x^2)(1+e^x)} = \int_0^1 \frac{dx}{1+x^2} = \operatorname{arctg} x \Big|_0^1 = \frac{\pi}{4}.$$

$$-1 \quad (1+x^2)\cdot(1+e^x) \quad \int_{-1} \quad \int_0 \quad \int_0 \Big| (1+x^2)(1+e^x) \qquad (1+x^2)(1+e^x) \Big| \; = \int_0 \frac{}{(1+x^2)(1+e^x)} = \int_0 \frac{}{1+x^2} = \operatorname{arctg} x \Big|_0 = \frac{\pi}{4}.$$

$t=-x$

$dt=-dx$

x	-1	0
t	1	0

$$\int_{-1}^{0} \frac{dx}{(1+x^2)(1+e^x)} = \int_{1}^{0} \frac{-dt}{(1+t^2)(1+e^{-t})} \overset{\cdot e^t}{=} \int_{0}^{1} \frac{dt\cdot e^t}{(1+t^2)(e^t+1)} = \int_{0}^{1} \frac{e^x\,dx}{(1+x^2)(e^x+1)}$$

(3) f непрекидна на $[-1,1]$. Доказати:

а) $\int_0^{\pi/2} f(\sin x)\,dx = \int_0^{\pi/2} f(\cos x)\,dx$

б) $\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$

в) Израчунати $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$

$\int_0^{\pi/2}$ лако!

а) $$\int_0^{\pi/2} f(\sin x)\,dx = \int_{\pi/2}^{0} f(\cos t)(-dt) = \int_0^{\pi/2} f(\cos t)\,dt \quad ✓$$

$x=\frac{\pi}{2}-t$, $t=\frac{\pi}{2}-x$, $dx=-dt$

$\sin x=\sin\left(\frac{\pi}{2}-t\right)=\cos t$

x	0	$\pi/2$
t	$\pi/2$	0

$\int_0^{\pi}$ лако!

б) $$\int_0^{\pi} x f(\sin x)\,dx = \int_{\pi}^{0} (\pi-t)\cdot f(\sin t)(-dt) = \int_0^{\pi} (\pi-t)\cdot f(\sin t)\,dt = \int_0^{\pi} \pi\cdot f(\sin t)\,dt - \int_0^{\pi} t\cdot f(\sin t)\,dt$$

$x=\pi-t$, $t=\pi-x$, $dx=-dt$

$\sin x=\sin(\pi-t)=\sin t$

x	0	π
t	π	0

$$\Rightarrow \int_0^{\pi} t\cdot f(\sin t)\,dt = \frac{1}{2}\int_0^{\pi} \pi f(\sin t)\,dt = \frac{\pi}{2}\int_0^{\pi} f(\sin t)\,dt$$

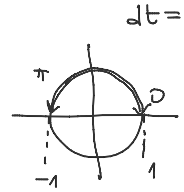
в) $$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^{\pi} \frac{x\sin x}{2-\sin^2 x}\,dx = \int_0^{\pi} x\cdot f(\sin x)\,dx \overset{б)}{=} \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx = \frac{\pi}{2}\cdot\int_0^{\pi} \frac{\sin x}{2-\sin^2 x}\,dx =$$

$f(x)=\frac{x}{2-x^2}$

$\cos x=t$

$dt=-\sin x\,dx$

x	0	π
t	1	-1



$$\pi\int^{-1} \frac{-dt}{} \qquad \pi\int^{1} dt \qquad \pi \qquad \pi^2$$

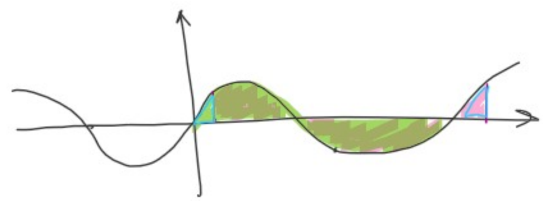
$$= \frac{\pi}{2}\int_{1}^{-1} \frac{-dt}{1+t^2} = \frac{\pi}{2}\cdot\int_{-1}^{1}\frac{dt}{1+t^2} = \frac{\pi}{2}\operatorname{arctg} t\,\Big|_{-1}^{1} = \frac{\pi}{2}\cdot\left(\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right) = \frac{\pi^2}{4}.$$

(*) $f:\mathbb{R}\to\mathbb{R}$, f непр. f периодична со периодот T

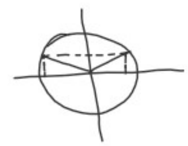
$$\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx$$

пр. $$\int_{\pi/14}^{29\pi/14} \sin x\,dx = \int_0^{2\pi}\sin x\,dx$$

$T = 2\pi$



(4) $$\int_0^{\pi} \underbrace{\frac{\sin^3 x\cdot\cos x}{1+\cos^4 x}}_{f(x)}\,dx = \int_0^{\pi/2} + \int_{\pi/2}^{\pi} = \int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^{0} f(x)\,dx = 0$$



$x = \pi - t$, $t = \pi - x$, $dx = -dt$

x	$\pi/2$	π
t	$\pi/2$	0

$\sin x = \sin(\pi - t) = \sin t$

$\cos x = \cos(\pi - t) = -\cos t$

$$\int_{\pi/2}^{\pi}\frac{\sin^3 x\cdot\cos x}{1+\cos^4 x}\,dx = \int_{\pi/2}^{0}\frac{\sin^3 t\cdot(-\cos t)}{1+\cos^4 t}(-dt) = \int_{\pi/2}^{0}\frac{\sin^3 t\cos t}{1+\cos^4 t}\,dt$$

(5) $$I_n = \int_0^{\pi/2}\sin^n x\,dx, \quad n\geq 2$$

$$I_n = \left|\begin{array}{ll} u = \sin^{n-1}x & du = (n-1)\cdot\sin^{n-2}x\cdot\cos x \\ dv = \sin x\,dx & v = -\cos x \end{array}\right| =$$

$$= \overbrace{-\cos x\cdot\sin^{n-1}x\,\Big|_0^{\pi/2}}^{0} + \int_0^{\pi/2}(n-1)\cdot\sin^{n-2}x\cdot\underbrace{\cos^2 x}_{1-\sin^2 x} =$$

$$= (n-1)\cdot\int_0^{\pi/2}\left(\sin^{n-2}x - \sin^n x\right)dx = (n-1)\cdot\left(I_{n-2} - I_n\right)$$

$$n\cdot I_n = (n-1)\cdot I_{n-2} \Rightarrow I_n = \frac{n-1}{n}\cdot I_{n-2}$$

$$I_{2k} = \frac{2k-1}{2k}\cdot I_{2k-2} = \frac{2k-1}{2k}\cdot\frac{2k-3}{2k-2}\cdot\ldots\cdot\frac{1}{2}\cdot I_0 = \frac{(2k-1)!!}{(2k)!!}\cdot I_0 = \frac{(2k-1)!!}{(2k)!!}\cdot\frac{\pi}{2}$$

$$I_{2k}=\frac{2k-1}{2k}\cdot I_{2k-2}=\frac{2k-1}{2k}\cdot\frac{2k-3}{2k-2}\cdot\ldots\cdot\frac{1}{2}\cdot I_0=\frac{(2k-1)!!}{(2k)!!}\cdot I_0=\frac{(2k-1)!!}{(2k)!!}\cdot\frac{\pi}{2}$$

$$I_0=\int_0^{\pi/2}\sin^0 x\,dx=\frac{\pi}{2}$$

$$I_{2k+1}=\frac{2k}{2k+1}\cdot\frac{2k-2}{2k-1}\cdot\ldots\cdot\frac{2}{3}\cdot I_1=\frac{(2k)!!}{(2k+1)!!}\cdot 1$$

$$I_1=\int_0^{\pi/2}\sin x\,dx=-\cos x\Big|_0^{\pi/2}=1$$

⑥ $$\underset{\overset{\|}{I}}{\int_0^{\pi}e^x\cdot\cos^2 x\,dx}=\int_0^{\pi}e^x\frac{1+\cos 2x}{2}dx=\underbrace{\int_0^{\pi}\frac{e^x}{2}dx}_{\frac{e^x}{2}\Big|_0^{\pi}}+\frac{1}{2}\int_0^{\pi}e^x\cos 2x\,dx=\frac{e^{\pi}-1}{2}+\frac{1}{2}\cdot\int_0^{\pi}e^x\cos 2x\,dx$$

домашно: приложете 2 парциално

$$e^{2xi}+e^{-2xi}=\left(\cos(2x)+\cancel{i\sin(2x)}\right)+\left(\cos(-2x)+\cancel{i\sin(-2x)}\right)=2\cos(2x)$$

$$\int_0^{\pi}e^x\cdot\frac{e^{2ix}+e^{-2ix}}{2}dx=\frac{1}{2}\cdot\int_0^{\pi}e^{(2i+1)x}dx+\frac{1}{2}\int_0^{\pi}e^{(1-2i)x}dx=$$

$$=\frac{1}{2}\cdot\frac{1}{2i+1}\cdot e^{(2i+1)x}\Big|_0^{\pi}+\frac{1}{2}\cdot\frac{1}{1-2i}\cdot e^{(1-2i)x}\Big|_0^{\pi}=$$

$$e^{2i\pi}=1$$

$$=\frac{1}{4i+2}\cdot\left(e^{(2i+1)\pi}-1\right)+\frac{1}{2-4i}\cdot\left(e^{(1-2i)\pi}-1\right)=$$

$$=\frac{1}{4i+2}\left(e^{\pi}-1\right)+\frac{1}{2-4i}\cdot\left(e^{\pi}-1\right)=\left(e^{\pi}-1\right)\cdot\frac{4}{4+16}=\frac{e^{\pi}-1}{5}.$$

$$I=\left(e^{\pi}-1\right)\cdot\left(\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{5}\right)=\frac{3}{5}\left(e^{\pi}-1\right).$$

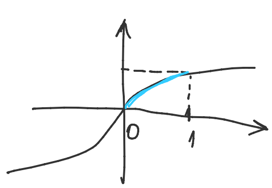
⑤ $$\int_0^1\frac{dx}{(x+1)\sqrt{x^2+1}}=I$$

домашно: Ойлерова смяна $\sqrt{x^2+1}=t-x$

$x=\operatorname{tg} t$

$t=\operatorname{arctg} x$

x	0	1
t	0	$\pi/4$



$$\sqrt{x^2+1}=\sqrt{\operatorname{tg}^2 t+1}=\sqrt{\frac{\sin^2 t+\cos^2 t}{\cos^2 t}}=$$

$$=\frac{1}{\sqrt{\cos^2 t}}=\frac{1}{\cos t}$$

$$I=\int_0^{\pi/4}\frac{dt}{\ldots}$$

x	0	1
t	0	$\pi/4$

$$dx = \frac{dt}{\cos^2 t}$$

$$I = \int_0^{\pi/4} \frac{\frac{dt}{\cancel{\cos^2} t}}{(\operatorname{tg} t + 1)\cdot \frac{1}{\cancel{\cos t}}} =$$

$$= \frac{1}{\sqrt{\cos^2 t}} = \frac{1}{\cos t}$$

$t \in [0, \pi/4]$

$\cos t > 0$

$$= \int_0^{\pi/4} \frac{dt}{\sin t + \cos t} = \frac{1}{\sqrt{2}} \cdot \int_0^{\pi/4} \frac{dt}{\sin\left(t + \frac{\pi}{4}\right)} =$$

$$\sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}(\sin x + \cos x)$$

$$= \frac{1}{\sqrt{2}} \int_{\pi/4}^{\pi/2} \frac{du}{\sin u} = \frac{1}{\sqrt{2}} \cdot \int_{\pi/4}^{\pi/2} \frac{\sin u \, du}{1 - \cos^2 u} = \frac{1}{\sqrt{2}} \cdot \int_{1/\sqrt{2}}^{0} \frac{-dv}{1 - v^2} = \frac{1}{\sqrt{2}} \cdot \int_0^{1/\sqrt{2}} \frac{dv}{(1-v)(1+v)} =$$

$t + \frac{\pi}{4} = u$

$dt = du$

t	0	$\pi/4$
u	$\pi/4$	$\pi/2$

$\cos u = v$

$-\sin u \, du = dv$

u	$\pi/4$	$\pi/2$
v	$\frac{1}{\sqrt{2}}$	0

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{2} \log \frac{1+v}{1-v} \Big|_0^{1/\sqrt{2}} =$$

$$= \frac{1}{2\sqrt{2}} \cdot \log \frac{1+\sqrt{2}}{\sqrt{2}-1} =$$

$$= \frac{1}{\sqrt{2}} \log(1+\sqrt{2}).$$

(6) $$\int_{1/a}^{a} \frac{\log x}{1+x^2}\, dx, \quad a > 1$$

$$= \int_{1/a}^{1} + \int_1^a$$

$$\int_{1/a}^{1} \frac{\log x}{1+x^2}\, dx = \Big/ \; \frac{1}{x} = t, \quad x = \frac{1}{t}, \quad dx = -\frac{dt}{t^2} \; \Big/$$

x	$1/a$	1
t	a	1

$$= \int_a^1 \frac{\log\left(\frac{1}{t}\right)}{1 + \frac{1}{t^2}} \cdot \left(-\frac{dt}{t^2}\right) = \int_a^1 \frac{-\log t}{\frac{t^2+1}{\cancel{t^2}}} \left(-\frac{dt}{\cancel{t^2}}\right) =$$

$$= \int_a^1 \frac{\log t}{1+t}\, dt$$

$$I = \int_{1/a}^{1} + \int_1^a = \int_a^1 f(x)\,dx - \int_1^a f(x)\,dx = 0.$$

думати: парцијални $\left[\begin{array}{l} u = \log x \\ dv = \frac{dx}{1+x^2} \end{array}\right] \rightsquigarrow \left[\begin{array}{l} t = \frac{1}{x} \\ \text{смена} \end{array}\right] \rightsquigarrow \left[\begin{array}{l} \text{парцијална} \\ u = \operatorname{arctg} \frac{1}{t} \end{array}\right]$ (напорно)