Петак 09.04.2021.

(1) $$\int \sin(\log x)\,dx = \left/ \begin{matrix} u=\sin(\log x) & dv=dx \\ du=\cos(\log x)\cdot\frac{1}{x}dx & v=x \end{matrix} \right/ = \sin(\log x)\cdot x - \int \cancel{x}\cdot\cos(\log x)\cdot\frac{1}{\cancel{x}}dx = \left/ \begin{matrix} u=\cos(\log x) & dv=dx \\ du=-\sin(\log x)\cdot\frac{dx}{x} & v=x \end{matrix} \right/ =$$

$$= \sin(\log x)\cdot x - \left(\cos(\log x)\cdot x - \int \cancel{x}\cdot\left(-\sin(\log x)\cdot\frac{dx}{\cancel{x}}\right)\right) = \sin(\log x)\cdot x - \cos(\log x)\,x - \int \sin(\log x)\,dx$$

$$\Rightarrow \int \sin(\log x)\,dx = \frac{x}{2}\left(\sin(\log x) - \cos(\log x)\right) + C$$

(2) $$\int \frac{\arcsin x}{x^2}\cdot\frac{1+x^2}{\sqrt{1-x^2}}\,dx = \left/ \begin{matrix} t=\arcsin x \\ dt=\frac{dx}{\sqrt{1-x^2}} \end{matrix} \right/ = \int \frac{t}{\sin^2 t}\cdot(1+\sin^2 t)\,dt = \int t\left(\frac{1}{\sin^2 t}+1\right)dt = \frac{t^2}{2} + \int \frac{t\,dt}{\sin^2 t} =$$

$t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$x=\sin t$

$\cos t>0$

$$= \left/ \begin{matrix} u=t & dv=\frac{dt}{\sin^2 t} \\ du=dt & v=-\mathrm{ctg}\,t \end{matrix} \right/ = \frac{t^2}{2} - t\,\mathrm{ctg}\,t + \int \mathrm{ctg}\,t\,dt = \ldots = \frac{t^2}{2} - t\,\mathrm{ctg}\,t + \log|\sin x| + C$$

$$\frac{\cos t}{\sin t} = \frac{\cos t\cdot\sin t}{1-\cos^2 t}$$

$u=\cos t$ ...

(3) $$I = \int \frac{dx}{\sin x\cdot\sqrt[4]{\cos 2x}} = \int \frac{\sin x\,dx}{(1-\cos^2 x)\cdot\sqrt[4]{2\cos^2 x-1}} = \left/ \begin{matrix} \cos x = t \\ dt=-\sin x\,dx \end{matrix} \right/ = -\int \frac{dt}{(1-t^2)\cdot\sqrt[4]{2t^2-1}} = -\int \frac{\frac{dt}{t}}{(1-t^2)\sqrt[4]{\frac{2}{t^2}-\frac{1}{t^4}}}$$

$\cos 2x = 2\cos^2 x - 1$

за: $x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right), t>0$

смена: $u=\sqrt[4]{\frac{2}{t^2}-\frac{1}{t^4}}$

$$u^4 = \frac{2}{t^2}-\frac{1}{t^4}$$

$$4u^3\,du = \left(-4\frac{1}{t^3}+4\frac{1}{t^5}\right)dt \Rightarrow \left(\frac{1}{t^5}-\frac{1}{t^3}\right)dt = u^3\,du$$

$$\frac{\frac{1}{t}dt}{1-t^2} = \frac{\left(\frac{1}{t^5}-\frac{1}{t^3}\right)dt}{(t-t^3)\cdot\left(\frac{1}{t^5}-\frac{1}{t^3}\right)} = \frac{u^3\,du}{\frac{1}{t^4}-\frac{1}{t^2}-\frac{1}{t^2}+1} = \frac{u^3\,du}{1-u^4}$$

$$\Rightarrow I = -\int \frac{u^3\,du}{(1-u^4)\,u} = \int \frac{u^2\,du}{u^4-1} = \ldots$$

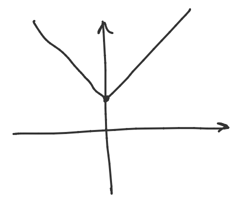
(4) (ф-ја која нема примитивну)

$f(x)=\mathrm{sgn}\,x$ на $\mathbb{R}$

$$F(0)=\lim_{x\to 0} F(x) = \lim_{x\to 0} C = C$$

$F'(x) = \operatorname{sgn} x \quad , F$-диф.

$$F(x) = \begin{cases} x + C, & x > 0 \\ -x + C, & x \leq 0 \end{cases}$$



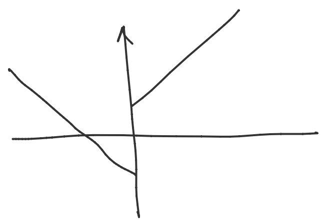
$1^\circ \; x > 0 \;, \; F'(x) = 1 \Rightarrow F = x + C_1$

$2^\circ \; x < 0 \;, \; F'(x) = -1 \Rightarrow F = -x + C_2$

F-диф $\Rightarrow F$-непр. $\lim\limits_{x \to 0+} F(x) = \lim\limits_{x \to 0-} F(x)$

$$\Downarrow$$

$$C_1 = C_2 = C \in \mathbb{R}$$



F - није диф. у 0, јер:

$$\left. \begin{array}{l} F'_+(0) = \lim\limits_{x \to 0+} F'(x) = 1 \\ \qquad\qquad\qquad\qquad \neq \\ F'_-(0) = \lim\limits_{x \to 0-} F'(x) = -1 \end{array} \right\} \lightning$$

⑤ (прекидна фја која има примитивну)

$f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 2x \sin\left(\frac{1}{x^2}\right) - \frac{2\cos\left(\frac{1}{x^2}\right)}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

$$F(x) = \begin{cases} x^2 \left(\sin \frac{1}{x^2}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

$$\left. \begin{array}{l} F'(x) = f(x) \quad , x \neq 0 \quad (\text{израчунати}) \\ F'(0) = \lim\limits_{h \to 0} \frac{F(h)}{h} = \lim\limits_{h \to 0} \left(h \cdot \sin \frac{1}{h^2}\right) = 0 = f(0) \end{array} \right\} (\forall x) \; F' = f \Rightarrow F \text{ је примитивна}$$

f није непр., јер: (у 0) $\quad f\left(\frac{1}{\sqrt{2n\pi}}\right) = 2\underbrace{\sin(2n\pi)}_{0} - \frac{2\cos(2n\pi)}{\frac{1}{\sqrt{2n\pi}}} = -2\sqrt{2n\pi} \not\to 0 \;\; n \to \infty$

(*) (касније) Свака непр. фја има примитивну.

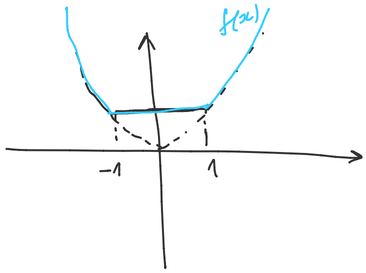
⑥ Наћи примитивне на $\mathbb{R}$:

а) $f(x) = \max\{1, x^2\}$

б) $f(x) = x|x|$

в) $f(x) = e^{-|x|}$

д)



f – непр.

$F = ?$

$1^\circ \quad |x|<1: \quad F'(x)=1 \Rightarrow F(x)=x+c_1$

$2^\circ \quad x>1 \ (x<-1): \quad F'(x)=x^2 \Rightarrow F(x)=\frac{x^3}{3}+c_2 \quad \left(\frac{x^3}{3}+c_3\right)$

F непр. у -1 и 1:

$$-\frac{1}{3}+c_3=-1+c_1 \quad , \quad 1+c_1=\frac{1}{3}+c_2$$

$c_1=c\in\mathbb{R}$

$c_2=\frac{2}{3}+c$

$c_3=-\frac{2}{3}+c$

$$F(x)=\begin{cases} \frac{x^3}{3}-\frac{2}{3}+c, & x<-1 \\ x+c, & -1\le x\le 1 \\ \frac{x^3}{3}+\frac{2}{3}+c, & x>1 \end{cases}$$

Још да ли $F'(1)=f(1)$, $F'(-1)=f(-1)$?

$F'(1)=f(1)$:

$$\left.\begin{aligned} F'_+(1)&=\lim_{x\to1+}F'(x)=\lim_{x\to1+}f(x)=f(1) \\ F'_-(1)&=\lim_{x\to1-}F'(x)=\lim_{x\to1-}f(x)=f(1) \end{aligned}\right\} \checkmark$$

Редови

$a_1, a_2, a_3, \dots$ низ бројева (из $\mathbb{C}$)

$S_1=a_1$

$S_n=a_n+S_{n-1}=a_n+a_{n-1}+\dots+a_1$

$\sum_{n=1}^{\infty}a_n$ конвергира $\Leftrightarrow \exists\lim_{N\to\infty}S_N \Leftrightarrow \exists\lim_{N\to\infty}\sum_{n=1}^{N}a_n\in\mathbb{R}$

a_n – општи члан реда

S_n – парцијална сума

$\sum_{n=1}^{\infty}a_n$ – ред (сума реда)

(a_n, S_n) – ред

⊛ $\sum\limits_{n=1}^{\infty} a_n$ конв $\Rightarrow \lim\limits_{n\to\infty} a_n = 0$

⊛ $\sum\limits_{n=1}^{\infty} a_n$ конв $\Rightarrow \lim\limits_{k\to\infty} \sum\limits_{n=k}^{\infty} a_n = 0$

пр. (геометријски ред)

$z \in \mathbb{C}$, $\underline{|z|<1}$ $\quad \sum\limits_{n=0}^{\infty} z^n$ конвергира и $\sum\limits_{n=0}^{\infty} z^n = \frac{1}{1-z}$

$$S_N = 1 + z + z^2 + \ldots + z^N \;/\cdot(1-z)$$

$$(1-z)\cdot S_N = (1-z) + (z-z^2) + (z^2-z^3) + \ldots + (z^N - z^{N+1}) = 1 - z^{N+1} \Rightarrow S_N = \frac{1-z^{N+1}}{1-z}$$

$$\lim_{N\to\infty} S_N = \frac{1}{1-z} - \frac{1}{1-z}\cdot \lim_{N\to\infty} z^{N+1} = \frac{1}{1-z} \Rightarrow \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \; |z|<1.$$

$$\lim_{N\to\infty} |z^{N+1}| = \lim_{N\to\infty} \underbrace{|z|}_{<1}{}^{N+1} = 0 \Rightarrow \lim_{N\to\infty} z^{N+1} = 0.$$

(1) Испитати конвергенцију редова:

а) $\sum \frac{1}{n^2-1}$ $\qquad$ в) $\sum \frac{2^n}{n}$

б) $\sum \log\left(1+\frac{1}{n}\right)$ $\qquad$ г) $\sum \frac{2n+1}{n^2(n+1)}$

$\sum = \sum\limits^{\infty}$?

принцип локализације:

$\sum\limits_{k=1}^{\infty} a_n$ конв. $\Leftrightarrow \sum\limits_{k=p}^{\infty} a_n$ конв.

$p \in \mathbb{N}$

а) $$\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \sum_{n=2}^{\infty} \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$$

$$\frac{1}{n^2-1} = \frac{1}{2}\left(\underbrace{\frac{1}{n-1} - \frac{1}{n+1}}\right)$$

$$\frac{(n+1)-(n-1)}{(n-1)(n+1)} = \frac{2}{n^2-1}$$

$$S_N = \sum_{n=2}^{N} \frac{1}{n^2-1} = \frac{1}{2}\sum_{n=2}^{N}\left(\frac{1}{n-1} - \frac{1}{n+1}\right) = \frac{1}{2}\Big(\underbrace{1 - \cancel{\tfrac{1}{3}}}_{2} + \underbrace{\tfrac{1}{2} - \cancel{\tfrac{1}{4}}}_{3} + \underbrace{\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{5}}}_{4} + \underbrace{\cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{6}}}_{5} + \ldots + \underbrace{\cancel{\tfrac{1}{N-3}} - \cancel{\tfrac{1}{N-1}}}_{N-2} + \underbrace{\cancel{\tfrac{1}{N-2}} - \tfrac{1}{N}}_{N-1} + \underbrace{\cancel{\tfrac{1}{N-1}} - \tfrac{1}{N+1}}_{N}\Big)$$

$$= \frac{1}{2}\left(1 + \frac{1}{2} - \underbrace{\frac{1}{N} - \frac{1}{N+1}}\right) \xrightarrow[N\to\infty]{} \frac{1}{2}\left(1+\frac{1}{2}\right) = \frac{3}{4}.$$

$\to 0$

$\Rightarrow$ ред конв.

б) $\sum a_n$ конв. $\Rightarrow a_n \to 0$

$$\boxed{a_n \not\to 0 \Rightarrow \sum a_n \text{ дивергира}}$$

$$\log\left(1+\frac{1}{n}\right) \xrightarrow[n\to\infty]{} \log 1 = 0 \not\Rightarrow \sum \log\left(1+\frac{1}{n}\right) \text{ конв.}$$

$$S_N = \sum_{n=1}^{N} \underbrace{\log\left(1+\frac{1}{n}\right)}_{= \log\frac{n+1}{n}} = \sum_{n=1}^{N}\left(\log(n+1)-\log n\right) = (\cancel{\log 2} - \overset{0}{\log 1}) + (\cancel{\log 3} - \cancel{\log 2}) + \cancel{\dots} + (\cancel{\log N} - \cancel{\log (N-1)}) + (\log(N+1) - \cancel{\log N})$$

$$= \log(N+1) \xrightarrow[N\to\infty]{} \infty \Rightarrow \text{ред дивергира}$$

в) $a_n = \frac{2^n}{n}$

$a_n \xrightarrow[n\to\infty]{} \infty \Rightarrow a_n \not\to 0 \Rightarrow \sum a_n$ див.

$\lim_{x\to\infty} \frac{2^x}{x} = \infty$ (Лопитал)

г) $\frac{2n+1}{n^2(n+1)} = \frac{A}{n} + \frac{B}{n^2} + \frac{C}{n+1}$

$\vdots$

(*) $\left.\begin{array}{l}\sum a_n \text{ конв.}\\ \sum b_n \text{ конв.}\end{array}\right\} \Rightarrow \sum \underbrace{(a_n+b_n)}_{c_n}$ конв.

$\alpha, \beta \in \mathbb{R}$

$\sum(\alpha a_n + \beta b_n)$ конв.

$\sum c_n$ конв. $\quad c_n = a_n + b_n$

(*) $\left.\begin{array}{l}\sum a_n \text{ конв.}\\ \sum b_n \text{ див.}\end{array}\right\} \quad \sum(a_n+b_n)$ див.

(ппс $\sum(a_n+b_n)$ конв. $\Rightarrow \underbrace{\sum((a_n+b_n)-a_n)}_{\sum b_n}$ конв. $\lightning$)