Понедељак 08.03.2021.

① Доказати да једначина $x^3-6x^2-1=0$ има тачно једно решење у скупу $\mathbb{R}$.

Циљ: скицирамо како изгледа

$f(x)=x^3-6x^2-1 \rightarrow$ диференцијабилна

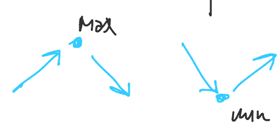
$f'(x)=3x^2-12x = 3x(x-4)$

$f'>0$, $x>4 \lor x<0$

$f'<0$, $0<x<4$

$f'=0$, $x\in\{0,4\}$

	$-\infty$	0	4	$+\infty$
f'	+	−	+	
f	↗	↘	↗	

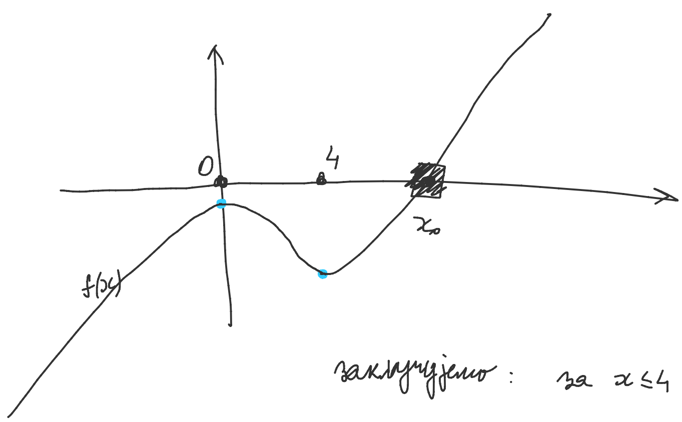


$f(0)=-1$

$f(4)=-33$

$\lim_{x\to-\infty} f(x)=-\infty$

$\lim_{x\to+\infty} f(x)=+\infty$



заључујемо: за $x\le 4$: $f(x)<0$

по Т о међувр: $f(4)<0$, „$f(\infty)$“ >0 $\Big\}$ $(\exists x_0)\ f(x_0)=0$

Зашто је x_0 јединствено? $f'>0 \Rightarrow f$ је строго ↗

$f(x)>f(y)$, $x>y$ $\Rightarrow$ само једна нула!

② Доказати неједнакости:

а) $\frac{2x}{\pi}\geq \sin x$, за $x\in\left[0,\frac{\pi}{2}\right]$

б) $e^x \geq 1+x+\frac{x^2}{2}+\frac{x^3}{6}$, $x\in\mathbb{R}$

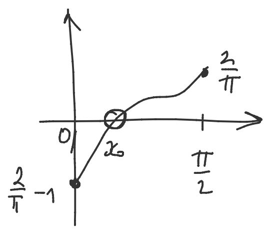
в) $\log(1+2x) > 2x - 2x^2$, $x>0$, доказати

а) $f(x) = \frac{2x}{\pi} - \sin x$ $(\geq 0\ ?)$

$f'(x) = \frac{2}{\pi} - \cos x$

f' непр и $f'(0)<0$

$f'(\frac{\pi}{2})>0$



$\Rightarrow (\exists x_0)\ f'(x_0)=0$, $x_0 \in (0, \frac{\pi}{2})$ ⟶ могли смо рећи $x_0 = \arccos\frac{2}{\pi}$.

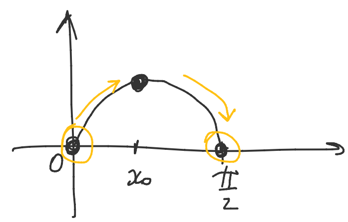
Постоји тачно једна, јер f' строго расте (јер и таква $-\cos x$)

на $(0,x_0)$: $f'<0 \Rightarrow f\searrow$

$(x_0, \frac{\pi}{2})$: $f'>0 \Rightarrow f\nearrow$

$f(0)=0$

$f(\frac{\pi}{2}) = \frac{2}{\pi}\cdot\frac{\pi}{2} - 1 = 0$



$\Rightarrow f(x) \geq 0, \forall x \in [0, \frac{\pi}{2}]$

$\Rightarrow \frac{2x}{\pi} \geq \sin x$.

б) $f(x) = e^x - 1 - x - \frac{x^2}{2} - \frac{x^3}{6}$ $\Rightarrow f(0) = e^0 - 1 - 0 - 0 - 0 = 0$

$f'(x) = e^x - 1 - x - \frac{x^2}{2}$ $\Rightarrow f'(0) = e^0 - 1 - 0 - 0 = 0$

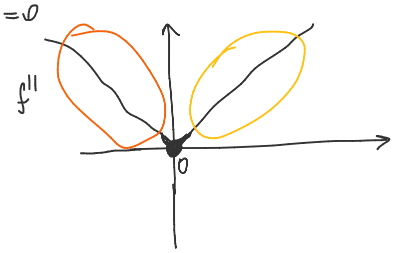
$f''(x) = e^x - 1 - x$ $\Rightarrow f''(0) = e^0 - 1 - 0 = 0$

$f'''(x) = e^x - 1$ $\Rightarrow f'''(0) = e^0 - 1 = 0$

$f'''>0$, $x>0$ $\Rightarrow f''\nearrow$

$f'''=0$, $x=0$ $\Rightarrow f''(0)=0$

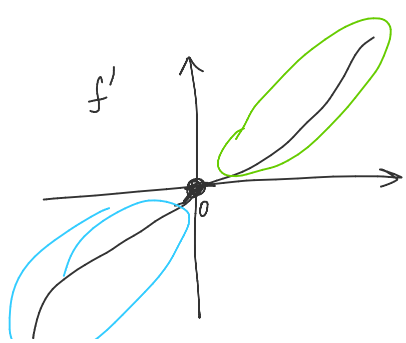
$f'''<0$, $x<0$ $\Rightarrow f''\searrow$



$\Rightarrow f''>0$, $x \in \mathbb{R}\setminus\{0\}$ $\Rightarrow f'\nearrow$ на $(-\infty,0) \cup (0,+\infty)$

$f''(0)=0$

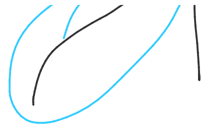
$f'(0)=0$

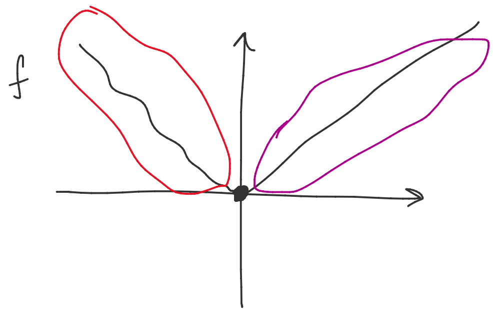


$\Rightarrow f'>0$, $x>0$ $\quad f\nearrow$

$f'<0$, $x<0$ $\Rightarrow f\searrow$

$f'=0$, $x=0$ $\quad f(0)=0$





$\Rightarrow f(x) \geq 0, \ f(x)=0 \Leftrightarrow x=0$

$\downarrow$

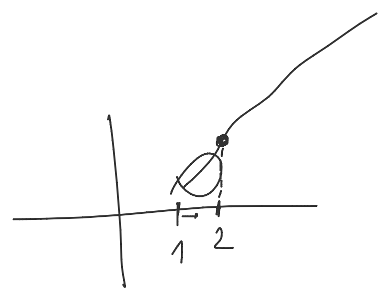
$e^x \geq 1+x+\frac{x^2}{2}+\frac{x^3}{6}$

③ f је диф. ф-ја $\quad \frac{1}{x} \leq f'(x) \leq x, \ \forall x>1$. Доказати

а) $\lim\limits_{x\to\infty} f(x)=\infty$

б) $\lim\limits_{x\to\infty} \frac{f(x)}{x^3}=0$

а) $\frac{1}{x} \leq f'(x)$ $\qquad (\log x)' = \frac{1}{x}$



$g(x)=f(x)-\log x$, g је диф. $\forall x>1$

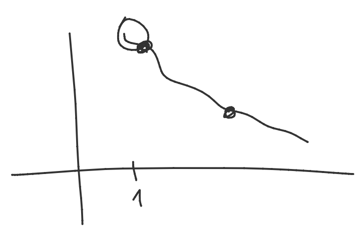
$g'(x) = f'(x) - \frac{1}{x} \geq 0 \ \Rightarrow g \nearrow$ на $(1,\infty) \ \Rightarrow g(x) \geq g(2) = f(2) - \log 2 \ , \forall x \geq 2$

$\Rightarrow f(x) \geq \log x + (f(2)-\log 2)$

$\lim\limits_{x\to\infty} \left(\log x + \underbrace{f(2)-\log 2}_{\text{const}}\right) = \infty \ \Rightarrow \lim\limits_{x\to\infty} f(x) = \infty.$

б) $h(x) = f(x) - \frac{x^2}{2}$

$h'(x) = f'(x) - x \leq 0 \ \Rightarrow h(x) \searrow \ \Rightarrow h(x) \leq f(2), \forall x \geq 2$



$\Rightarrow f(x) \leq f(2) + \frac{x^2}{2}$

$\lim\limits_{x\to\infty} \frac{f(x)}{x^3} \leq \lim\limits_{x\to\infty} \frac{f(2)+\frac{x^2}{2}}{x^3} = 0$ $\qquad \frac{1}{2x} \to 0 \qquad \to 0$

④ $x,y,a,b>0 \Rightarrow \left(\frac{x+y}{a+b}\right)^{x+y} \le \left(\frac{x}{a}\right)^{x}\cdot\left(\frac{y}{b}\right)^{y}$ $\left(\log u^{v} = v\cdot \log u\right)$

$\Updownarrow$ log монотон

$$(x+y)\cdot\log\left(\frac{x+y}{a+b}\right) \le x\cdot\log\left(\frac{x}{a}\right) + y\cdot\log\left(\frac{y}{b}\right)$$

y, a, b – схватимо као конст.

$f(x) = (x+y)\log\left(\frac{x+y}{a+b}\right) - x\log\left(\frac{x}{a}\right) - y\log\left(\frac{y}{b}\right)$ (хоћемо $f(x)\le 0$)

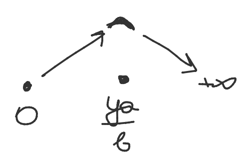
$$f'(x) = 1\cdot\log\left(\frac{x+y}{a+b}\right) + \cancel{(x+y)}\cdot\frac{1}{\frac{\cancel{x+y}}{\cancel{a+b}}}\cdot\frac{1}{\cancel{a+b}} - 1\cdot\log\left(\frac{x}{a}\right) - \cancel{x}\cdot\frac{1}{\frac{\cancel{x}}{\cancel{a}}}\cdot\frac{1}{\cancel{a}} - 0 =$$

$$= \log\left(\frac{x+y}{a+b}\right) + 1 - \log\left(\frac{x}{a}\right) - 1 =$$

$$= \log\left(\frac{(x+y)a}{(a+b)x}\right)$$

$f'>0 \Leftrightarrow \log(\;--\;)>0 \Leftrightarrow \frac{(x+y)a}{(a+b)x} > 1 \Leftrightarrow \cancel{x}a+ya > a\cancel{x}+bx \Leftrightarrow x < \frac{ya}{b} \Rightarrow f\nearrow$ на $\left(0, \frac{ya}{b}\right)$

$f'=0 \Leftrightarrow x = \frac{ya}{b}$



$f'<0 \Leftrightarrow x > \frac{ya}{b} \Rightarrow f\searrow$ на $\left(\frac{ya}{b}, +\infty\right)$

$\Rightarrow f(x) \le f\left(\frac{ya}{b}\right), \ \forall x>0$

$$f\left(\frac{ay}{b}\right) = \left(\frac{ay}{b}+y\right)\log\left(\frac{\frac{ay}{b}+y}{a+b}\right) - \frac{ay}{b}\cdot\log\left(\frac{y}{b}\right) - y\log\left(\frac{y}{b}\right) = \ldots = 0$$

$\Rightarrow f(x)\le 0$ ✓

I (Лајбницова формула)

f,g n-пута диф., тада је и $(f\cdot g)$ n-пута диф. и

$\left(\begin{array}{l}\exists f', f'', f''', \ldots, f^{(n-1)}, f^{(n)} \\ g', \ \ldots \ , g^{(n)}\end{array}\right)$ $(fg)^{(n)}(x) = \sum_{k=0}^{n}\binom{n}{k} f^{(k)}(x)\cdot g^{(n-k)}(x).$

$\left[(a+b)^{n} = \sum_{k=0}^{n}\binom{n}{k}a^{k}b^{n-k}\right]$

$n=1$: $(fg)' = \sum_{k=0}^{1}\binom{1}{k}f^{(k)}g^{(n-k)} = \underbrace{fg'}_{k=0} + \underbrace{f'g}_{k=1}$

$$n=1: \quad (fg)' = \sum_{k=0}^{1} \binom{1}{k} f^{(k)} g^{(1-k)} = \underbrace{fg'}_{k=0} + \underbrace{f'g}_{k=1}$$

⑤ Наћи n-ти извод ф-ја:

а) $h(x) = \dfrac{1}{x^2-3x+2}$

б) $h(x) = \sin^4 x + \cos^4 x$, домаћи $\left(\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x = 1 - 2\sin^2 x \cos^2 x\right)$

а) $x^2-3x+2 = x^2 - x - 2x + 2 = x(x-1) - 2(x-1) = (x-1)(x-2)$

$$h(x) = \frac{1}{x^2-3x+2} = \frac{1}{(x-1)(x-2)}, \qquad f(x) = \frac{1}{x-1}, \quad g(x) = \frac{1}{x-2}$$

$$f'(x) = -\frac{1}{(x-1)^2}, \quad f''(x) = \left(-(x-1)^{-2}\right)' = \left(-(-2)\cdot(x-1)^{-3}\right) = \frac{2}{(x-1)^3}, \quad f'''(x) = \frac{-6}{(x-1)^4}, \ \ldots$$

домаћи, индукцијом

$$f^{(n)}(x) = \frac{(-1)^n \cdot n!}{(x-1)^{n+1}}, \quad g^{(n)} = \frac{(-1)^n\, n!}{(x-2)^{n+1}}$$

$$h^{(n)}(x) = (f\cdot g)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} \cdot f^{(k)}(x) \cdot g^{(n-k)}(x) = \sum_{k=0}^{n} \frac{n!}{\cancel{k!}\cdot\cancel{(n-k)!}} \cdot \frac{(-1)^k \cancel{k!}}{(x-1)^{k+1}} \cdot \frac{(-1)^{n-k}\cancel{(n-k)!}}{(x-2)^{n-k+1}}$$

$$= \frac{(-1)^n \cdot n!}{(x-1)(x-2)} \cdot \sum_{k=0}^{n} \frac{1}{(x-1)^k \cdot (x-2)^{n-k}} = \frac{(-1)^n n!}{(x-1)(x-2)} \cdot \frac{\frac{1}{(x-1)^{n+1}} - \frac{1}{(x-2)^{n+1}}}{\frac{1}{x-1} - \frac{1}{x-2}}$$

$a = \frac{1}{x-1}, b = \frac{1}{x-2}$

$$a^{n+1} - b^{n+1} = (a-b)(a^n + a^{n-1}b + a^{n-2}b^2 + \ldots + a^2 b^{n-2} + a b^{n-1} + b^n) =$$

$$= (a-b)\cdot \sum_{k=0}^{n} a^k b^{n-k}$$

$$\Rightarrow \sum_{k=0}^{n} a^k b^{n-k} = \frac{a^{n+1}-b^{n+1}}{a-b}$$

Добије се: $h^{(n)}(x) = (-1)^n n! \cdot \left(\frac{1}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}} \right)$

Лопиталова правила

(Л'опиталова, L'Hôpital's rule)

$\frac{0}{0}$, $\frac{}{\infty}$

$\boxed{Т}$ $f, g : (a,b) \to \mathbb{R}$ диф. $g'(x) \neq 0 \ \forall x \in (a,b)$

Ако $\exists \lim\limits_{x \to a+} \frac{f'(x)}{g'(x)} = A$ и ако важи једна од ствари

$$\left.\begin{array}{l} 1^\circ \ \lim\limits_{x \to a+} f(x) = \lim\limits_{x \to a+} g(x) = 0 \\ \\ 2^\circ \ \lim\limits_{x \to a+} g(x) = \infty \end{array}\right\} \Rightarrow \exists \lim_{x \to a+} \frac{f(x)}{g(x)} = A.$$

(аналогно и за $b-$)

Пр. Како уз помоћ Лоп. да изр. $\lim\limits_{x \to \infty} \frac{\sin x}{x}$? $(b = \infty)$

2° важи, јер $x \xrightarrow[x \to \infty]{} \infty$

$$\lim_{x \to \infty} \frac{\sin x}{x} = \lim_{x \to \infty} \frac{\cos x}{1} = \lim_{x \to \infty} \cos x \ ? \Rightarrow$$ почетни не постоји!

НИЈЕ ДОБРО

$$\frac{\sin x}{x} \xrightarrow[x \to \infty]{} 0$$ он постоји!

(1) $\lim\limits_{x \to 0} \frac{x \cos x - \sin x}{x^3}$

Да ли $\exists \lim\limits_{x \to 0} \frac{f'}{g'}$?

$f(x) = x \cos x - \sin x$

$g(x) = x^3$

(1°) $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} g(x) = 0$

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{\cancel{\cos x} - x \sin x - \cancel{\cos x}}{3x^2} = \lim_{x \to 0} - \frac{\sin x}{3x} = -\frac{1}{3}$$

$$\overset{ЛП}{\Longrightarrow} \lim_{x \to 0} \frac{f(x)}{g(x)} = -\frac{1}{3}.$$