д) $x \arcsin x + \sqrt{1-x^2}$

ђ) $e^x \cdot \cos x$

д) I начин : развити $\arcsin x$ и $\sqrt{1-x^2}$ и сабрати ... довати

II начин : $f(x) = x\arcsin x + \sqrt{1-x^2}$

$$f'(x) = \arcsin x + x \cdot \frac{1}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) = \arcsin x$$

$$f''(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} \cdot (-x^2)^n = \ldots = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot x^{2n} \quad , |x|<1$$

$$f'(x) = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot \frac{x^{2n+1}}{2n+1} + \underline{\underline{c_1}}, |x|<1 \quad , \quad f'(0) = \arcsin 0 = 0 \Rightarrow c_1 = 0$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot \frac{x^{2n+2}}{(2n+2)(2n+1)} + \underline{\underline{c_2}}, \quad |x|<1 \quad , \quad f(0) = 1 \Rightarrow c_2 = 1$$

Да ли за $x = \pm 1$ ово важи?

↳ $x^{2n+2} = 1$

$$a_n = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{1}{(2n+2)(2n+1)} > 0 , \quad \frac{a_n}{a_{n+1}} = \ldots = 1 + \frac{5}{2n+1} + \frac{6}{(2n+1)^2}$$

$$\lim_{n\to\infty} n \cdot \left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{5}{2} > 1 \overset{\text{Рабе}}{\Longrightarrow} \text{конв} \overset{\text{Абел}}{\Longrightarrow} D = [-1,1].$$

ђ) $\underbrace{e^x}_{(--)} \cdot \underbrace{\cos x}_{(--)}$

Re$(e^x \cdot e^{ix})$

$$e^x \cdot e^{ix} = e^x \cdot (\cos x + i \sin x) = \underline{e^x \cos x} + i\, e^x \sin x$$

$\|$

$e^{x(1+i)}$

$$e^x \cos x = \mathrm{Re}\,(e^{x(1+i)}) = \mathrm{Re} \sum_{n=0}^{\infty} \frac{(x(1+i))^n}{n!} = \mathrm{Re} \sum_{n=0}^{\infty} \frac{x^n}{n!} \cdot (1+i)^n = \mathrm{Re} \sum_{n=0}^{\infty} \underbrace{\frac{x^n}{n!} \cdot 2^{n/2}}_{\in \mathbb{R}} \cdot \underbrace{e^{in\frac{\pi}{4}}}_{\in \mathbb{C}}$$

$$1+i = \sqrt{2} \cdot \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2} \cdot e^{i\frac{\pi}{4}} \Big/^{n} \Rightarrow (1+i)^n = 2^{n/2} \cdot e^{in\frac{\pi}{4}} = 2^{n/2} \cdot \left(\underline{\underline{\cos \frac{n\pi}{4}}} + i \sin \frac{n\pi}{4}\right)$$

$$e^x \cos x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \cdot 2^{n/2} \cdot \mathrm{Re}\left(e^{in\frac{\pi}{4}}\right) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \cdot 2^{n/2} \cdot \cos\frac{n\pi}{4} \quad , \underline{\underline{D = \mathbb{R}}}$$

$e^z = \sum^{\infty} z^n$ ... на $\mathbb{C}$ $\Rightarrow$ $\forall x \in \mathbb{R}$ важи

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \text{ важи на } \mathbb{C} \Rightarrow \forall x \in \mathbb{R} \text{ важи}$$

домаћи: а) шта је развој за $e^x \cdot \sin x$?

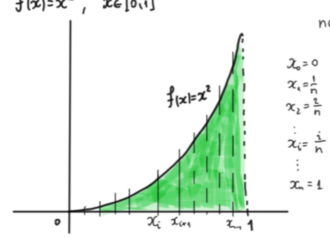
б) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n/2} \cdot \sin\frac{\pi n}{2}}{n!} \cdot \pi^n$

Одређени интеграл

Дефиниције са предавања:

МОТИВАЦИЈА: (површина испод графика)

$f(x)=x^2$, $x \in [0,1]$



подела је на истим растојањима (Кошијев интеграл)

P - површина испод графика ф-је (изнад x-осе)

$$P \approx \sum_{i=0}^{n-1} f(x_i) \cdot \frac{1}{n}$$

$$\boxed{P = \lim_{n \to +\infty} \sum_{i=0}^{n-1} f(x_i) \cdot \frac{1}{n}}$$

$$P \approx \sum_{i=0}^{n-1} (x_{i+1} - x_i) \cdot x_i^2$$

апроксимација

Риманов интеграл (општије)

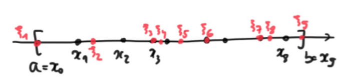
$[a,b] \subseteq \mathbb{R}$

затворен интервал $-\infty < a < b < +\infty$

Деф1: Подела P од $[a,b]$ је коначан скуп тачака $P = \{x_0, x_1, \dots, x_n\}$ из $[a,b]$ тј. $x_0 = a < x_1 < x_2 < \dots < x_{n-1} < x_n = b$.

$\Delta_i := [x_{i-1}, x_i]$, $i = \overline{1,n}$ - подеони интервали или интервали поделе P

Пар (P, ξ) је ПОДЕЛА СА ИСТАКНУТИМ ТАЧКАМА ОД $[a,b]$ ако је $P = \{x_0, x_1, \dots, x_n\}$ подела од $[a,b]$ и $\xi = (\xi_1, \dots, \xi_n)$ при чему је $\xi_i \in \Delta_i = [x_{i-1}, x_i]$.



Дијаметар поделе P је број $\lambda(P)$ који се дефинише као

$$\lambda(P) = \max_{1 \le i \le n} |x_i - x_{i-1}|$$

Деф2: Нека је (P, ξ) подела са истакнутим тачкама од $[a,b]$. Сума

$$\sigma(f; P, \xi) := \sum_{i=1}^{n} f(\xi_i) \cdot \Delta x_i$$

$\Delta x_i = x_{i+1} - x_i$

се назива Римановом сумом ф-је f по (P, ξ) на $[a,b]$.

Деф3: Кажемо да је I Риманов интеграл ф-је f на $[a,b]$ ако

$$(\forall \varepsilon>0)\ (\exists \delta>0)\ (\forall (P,\xi))$$
$$\lambda(P)<\delta \Rightarrow \left|\sigma(f;P,\xi)-I\right|<\varepsilon$$

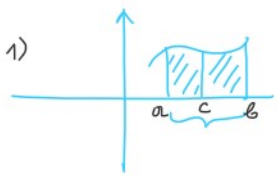
Деф4: Ф-ја $f:[a,b]\to \mathbb{C}$ је Риман интеграбилна ако постоји I из Деф3, тј. ако $\exists \lim\limits_{P} \sigma(f;P)$.

Ознака $I \equiv \int_a^b f(x)\,dx$ - интеграл ф-је f на затв. интервалу $[a,b]$

Особине:

1) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

1)



2) $\int_a^a f(x)\,dx = 0$

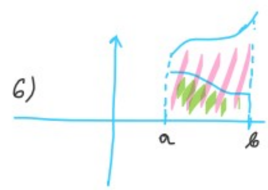
3) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

4) $\int_a^b (\lambda f(x)+\mu g(x))\,dx = \lambda\cdot\int_a^b f(x)\,dx + \mu\cdot\int_a^b g(x)\,dx$

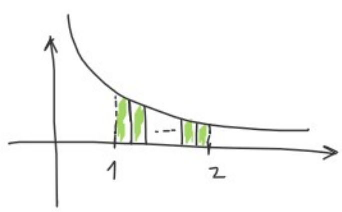
5) $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

6) $f(x)\le g(x),\ \forall x\in[a,b] \Rightarrow \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

6)



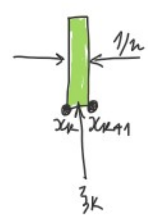
① Израчунати по деф. $\int_1^2 \frac{dx}{x^2}$



подела: $\mathcal{P}:\ 1=x_0<x_1<\dots<x_n=2$

$x_k = \frac{k}{n}+1,\ k=\overline{0,n}$

$\mathcal{P}:\ 1, \frac{n+1}{n}, \frac{n+2}{n}, \dots, \frac{2n-1}{n}, 2$



$\xi_k \in [x_k, x_{k+1}]$ - истакнуте тачке

$\xi_k = \sqrt{x_k\cdot x_{k+1}}$, $x_k<\xi_k<x_{k+1}$

$f(x)=\frac{1}{x^2}$

$$\sigma(f;\mathcal{P},\xi) = \sum_{k=0}^{n-1} f(\xi_k)\cdot(x_{k+1}-x_k) = \sum_{k=0}^{n-1}\frac{1}{x_k\cdot x_{k+1}}\cdot\frac{1}{n} = \sum_{k=0}^{n-1}\frac{n^2}{(k+n)(k+n+1)}\cdot\frac{1}{n} = n\cdot\sum_{k=0}^{n-1}\frac{1}{(k+n)(k+n+1)} =$$

$$= n\cdot \sum_{k=0}^{n-1}\left(\frac{1}{k+n} - \frac{1}{k+n+1}\right) = n\cdot\left(\frac{1}{n} - \frac{1}{2n}\right) = \frac{1}{2}.$$

$$\left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \dots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right)$$

T (Основна теорема интегралног рачуна) $f:[a,b]\to\mathbb{R}$ непр., онда је $F(x)=\int_a^x f(t)\,dt$ диференцијабилна и $F'(x)=f(x)$.

T (Њутн-Лајбницова формула) $F:[a,b]\to\mathbb{R}$ примитивна за непрекидну f, онда је

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b)-F(a).$$

пр. 1) $\int_1^2 \frac{dx}{x^2} = \left(-\frac{1}{x}\right)\Big|_1^2 = -\frac{1}{2} - \left(-\frac{1}{1}\right) = \frac{1}{2}.$

$f(x)=\frac{1}{x^2}$ непр. на $[1,2]$

$F(x) = -\frac{1}{x}$

2) $\int_0^1 \frac{dx}{1+x^2} = (\operatorname{arctg} x)\Big|_0^1 = \frac{\pi}{4} - 0 = \frac{\pi}{4}.$

T (Парцијална интеграција) $u,v\in C^1([a,b])$, онда $\int_a^b uv'\,dx = u\cdot v\Big|_a^b - \int_a^b vu'\,dx$

T (Смена променљиве) $f\in C[a,b]$ и $\varphi:[\alpha,\beta]\to[a,b]$, $\varphi\in C^1([\alpha,\beta])$, $\varphi(\alpha)=a$, $\varphi(\beta)=b$, тада

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\cdot\varphi'(t)\,dt$$

2) $$\int_0^{\sqrt3} x \operatorname{arctg} x\,dx = \left|\begin{matrix} u=\operatorname{arctg} x & du=\frac{dx}{x^2+1} \\ dv = x\,dx & v=\frac{x^2}{2}\end{matrix}\right| = \left(\frac{x^2}{2}\cdot \operatorname{arctg} x\right)\Big|_0^{\sqrt3} - \int_0^{\sqrt3} \frac{x^2}{2}\cdot\frac{dx}{x^2+1} =$$

$$=\left(\frac{3}{2}\cdot\frac{\pi}{3} - 0\right) - \frac{1}{2}\int_0^{\sqrt3}\left(1-\frac{1}{1+x^2}\right)dx = \frac{\pi}{2} - \frac{1}{2}\cdot\left(\int_0^{\sqrt3} dx - \int_0^{\sqrt3}\frac{dx}{1+x^2}\right) =$$

$$= \frac{\pi}{2} - \frac{1}{2}\left(x\Big|_0^{\sqrt3} - \operatorname{arctg} x\Big|_0^{\sqrt3}\right) = \frac{\pi}{2} - \frac{1}{2}\cdot\left(\sqrt3 - \frac{\pi}{3}\right) = \frac{2\pi}{3} - \frac{\sqrt3}{2}.$$

3) $$\int_0^a x^2\cdot\sqrt{a^2-x^2}\,dx = \left| x = a\sin t \in[0,a] \quad 0\to 0 \quad \begin{array}{c|c} x & t \\ \hline 0 & 0 \end{array}\right| =$$

(3) $\displaystyle\int_0^a x^2\cdot\sqrt{a^2-x^2}\,dx = \Bigg/ \begin{array}{l} x = a\sin t \in [0,a] \\ t = \arcsin\frac{x}{a} \in [0,\frac{\pi}{2}] \end{array} \quad \begin{array}{l} 0\to 0 \\ a \to \frac{\pi}{2} \end{array} \quad \begin{array}{c|c} x & t \\ \hline 0 & 0 \\ a & \frac{\pi}{2} \end{array} \Bigg/ =$

$a>0$

$t\in[0,\frac{\pi}{2}] \Rightarrow \cos t \geq 0$

у теореми:

$[\alpha,\beta]=[0,\frac{\pi}{2}]$

$[a,b]=[0,a]$

$\varphi'(t) = a\cos t$

$\varphi(t) = x(t) = a\sin t : [0,\frac{\pi}{2}] \to [0,a]$

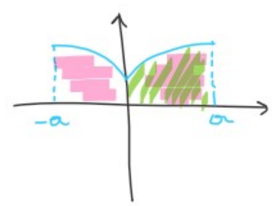
$$= \int_0^{\pi/2} (a\sin t)^2\cdot\sqrt{a^2-(a\sin t)^2}\cdot a\cos t\,dt = \int_0^{\pi/2} a^2\sin^2 t\cdot\overset{a\cos t}{\overset{\parallel}{\sqrt{a^2\cdot\cos^2 t}}}\cdot a\cos t\,dt =$$

$$= a^4\cdot\int_0^{\pi/2}\sin^2 t\cdot\cos^2 t\,dt = a^4\cdot\frac{1}{4}\int_0^{\pi/2}(\underbrace{2\sin t\cos t}_{\sin 2t})^2\,dt = \frac{a^4}{4}\cdot\int_0^{\pi/2}\frac{1-\cos 4t}{2}\,dt =$$

$$= \frac{a^4}{4}\cdot\int_0^{\pi/2}\frac{dt}{2} - \frac{a^4}{4}\cdot\int_0^{\pi/2}\frac{\cos 4t}{2}\,dt = \frac{a^4}{4}\cdot\left(\frac{t}{2}\right)\Big|_0^{\pi/2} - \frac{a^4}{4}\cdot\left(\frac{\sin 4t}{8}\right)\Big|_0^{\pi/2} =$$

$$= \frac{a^4}{4}\cdot\frac{\pi}{4} - \frac{a^4}{4}\cdot\frac{1}{8}\cdot(0-0) = \frac{a^4\pi}{16}.$$

(*) $f(x)$ – парна, онда $\displaystyle\int_{-a}^a f(x)\,dx = 2\cdot\int_0^a f(x)\,dx$



$$\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx = \int_a^0 f(-t)\,(-dt) + \int_0^a f(x)\,dx =$$

$x=-t$

$dx=-dt$

$0\to 0$

$-a\to a$

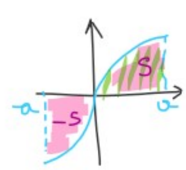
$$= \int_0^a f(-t)\,dt + \int_0^a f(x)\,dx =$$

$$= \int_0^a f(t)\,dt + \int_0^a f(x)\,dx = 2\cdot\int_0^a f(x)\,dx.$$

$f(-t)=f(t)$

Променљива није битна!

(*) $f(x)$ – непарна, онда $\displaystyle\int_{-a}^a f(x) = 0$ ← доказати



$-f(x) = f(-x)$

(4) $$\int_{-\pi/2}^{\pi/2}\frac{\sin^3 x + 2\sin x + \cos^3 x + 2\cos x + \sin x\cdot\log(\cos^2 x+7)}{(\cos^2 x+2)\cdot(\cos^2 x+3)}\,dx$$

$\parallel$

$f(x)$

$$g(x) = \frac{\sin^3 x + 2\sin x + \sin x \cdot \log(\cos^7 x + 7)}{(\cos^2 x + 2)(\cos^2 x + 3)}$$

$$h(x) = \frac{\cos^3 x + 2\cos x}{(\cos^2 x + 2)(\cos^2 x + 3)} = \frac{\cos x}{\cos^2 x + 3}$$

$g(-x) = -g(x)$ – нечетна

$h(x) = h(-x)$ – четна

$$\int_{-\pi/2}^{\pi/2} f(x)\,dx = \underbrace{\int_{-\pi/2}^{\pi/2} g(x)\,dx}_{0} + \int_{-\pi/2}^{\pi/2} h(x)\,dx = 2 \cdot \int_{0}^{\pi/2} h(x)\,dx = 2 \cdot \int_{0}^{\pi/2} \frac{\cos x\,dx}{\cos^2 x + 3} = \ldots$$

довърши

хинт: смяна $\sin x = t$

решение: $\frac{\log 3}{2}$.