2) $\int R(x, \sqrt{x^2+\lambda^2})\,dx$

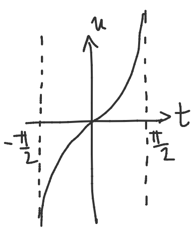
$x = \lambda \operatorname{tg} t$

① $$\int \frac{x}{\sqrt{3x^2-6x+14}}\,dx = \frac{1}{\sqrt{3}}\int \frac{x}{\sqrt{(x-1)^2+\frac{11}{3}}}\,dx = \frac{1}{\sqrt{3}}\int \frac{u+1}{\sqrt{u^2+\frac{11}{3}}}\,du = I$$

$x-1=u$

$dx=du$

$$3x^2-6x+14 = 3\left(x^2-2x+\frac{14}{3}\right) = 3\left((x-1)^2+\frac{11}{3}\right)$$

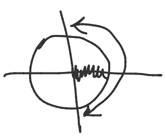


$\lambda^2 = \frac{11}{3}$, $\lambda = \sqrt{\frac{11}{3}}$

$$u = \sqrt{\frac{11}{3}}\cdot \operatorname{tg} t, \quad \sqrt{u^2+\frac{11}{3}} = \sqrt{\frac{11}{3}\operatorname{tg}^2 t + \frac{11}{3}} = \sqrt{\frac{11}{3}}\cdot\sqrt{\frac{\sin^2 t}{\cos^2 t}+\frac{\cos^2 t}{\cos^2 t}} = \sqrt{\frac{11}{3}}\cdot\frac{1}{|\cos t|} = \sqrt{\frac{11}{3}}\cdot\frac{1}{\cos t}$$

$u\in\mathbb{R} \Rightarrow \operatorname{tg} t\in\mathbb{R} \underset{\text{нпр.}}{\Rightarrow} t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \Rightarrow \cos t>0$

$$du = \sqrt{\frac{11}{3}}\cdot\frac{1}{\cos^2 t}\cdot dt$$



$$I = \frac{1}{\sqrt{3}}\int \frac{\sqrt{\frac{11}{3}}\operatorname{tg} t+1}{\cancel{\sqrt{\frac{11}{3}}}\cdot\cancel{\frac{1}{\cos t}}}\cdot\cancel{\sqrt{\frac{11}{3}}}\cdot\frac{1}{\cos^{\cancel{2}} t}\,dt = \frac{1}{\sqrt{3}}\int \frac{\sqrt{\frac{11}{3}}\operatorname{tg} t+1}{\cos t}\,dt = \int R(\sin t,\cos t)\,dt$$

$v = \operatorname{tg}\frac{t}{2} \Rightarrow$

$$\sin t = \frac{2v}{1+v^2}, \quad \cos t = \frac{1-v^2}{1+v^2}, \quad dt = \frac{2dv}{v^2+1}$$

$v^2=w$, $dw = 2v\,dv$

$$I = \frac{1}{\sqrt{3}}\cdot\int\frac{\sqrt{\frac{11}{3}}\cdot\frac{2v}{1-v^2}+1}{\frac{1-v^2}{\cancel{1+v^2}}}\cdot\frac{2dv}{\cancel{v^2+1}} = \frac{2}{\sqrt{3}}\cdot\int\frac{\sqrt{\frac{11}{3}}\cdot v\,dv}{(1-v^2)^2} + \frac{2}{\sqrt{3}}\cdot\int\frac{dv}{1-v^2} =$$

$$= \frac{\sqrt{11}}{3}\int\frac{dw}{(1-w)^2} + \frac{2}{\sqrt{3}}\int\frac{dv}{(1-v)(1+v)} = \frac{\sqrt{11}}{3}\left(-\frac{(1-w)^{-1}}{-1}\right) + \frac{1}{\sqrt{3}}\left(-\log|1-v|+\log|1+v|\right) + C =$$

$$\frac{1/2}{1-v} + \frac{1/2}{1+v}$$

$$= \frac{\sqrt{11}}{3}\,\frac{1}{1-w} + \frac{1}{\sqrt{3}}\log\left|\frac{1+v}{1-v}\right| + C = \frac{\sqrt{11}}{3}\,\frac{1}{1-v^2} + \frac{1}{\sqrt{3}}\log\left|\frac{1+v}{1-v}\right| + C =$$

$$= \frac{\sqrt{11}}{3}\,\frac{1}{1-\operatorname{tg}^2\frac{t}{2}} + \frac{1}{\sqrt{3}}\log\left|\frac{1+\operatorname{tg}\frac{t}{2}}{1-\operatorname{tg}\frac{t}{2}}\right| + C = \dots$$

$$= \frac{\sqrt{11}}{3}\,\frac{1}{1-tg^2\frac{t}{2}} + \frac{1}{\sqrt{3}}\log\left|\frac{1+tg\frac{t}{2}}{1-tg\frac{t}{2}}\right| + C = \ldots$$

$$t = arctg\left(\sqrt{\tfrac{3}{11}}\,u\right) = arctg\left(\sqrt{\tfrac{3}{11}}(x-1)\right)$$

2)⊛ $\int R(x, \sqrt{x^2+\lambda^2})\,dx = \int R(\lambda sh t, \lambda ch t)\,\lambda ch t\,dt = \int R_1(e^t, e^{-t})\,dt$

$x = \lambda \cdot sh t$ $\qquad sh^2 t + 1 = ch t$ $\qquad (e^t = u$ мр.$)$

$dx = \lambda ch t\,dt$

3) $\int R(x, \sqrt{x^2-\lambda^2})\,dx$

$$x = \frac{\lambda}{\cos t}$$

② $I = \int \frac{x-1}{\sqrt{x^2-4x+3}}\,dx = \int \frac{x-1}{\sqrt{(x-2)^2-1}}\,dx = \int \frac{\frac{1}{\cos t}+1}{|tg t|}\cdot tg t \cdot \frac{1}{\cos t}\,dt$

$x^2-4x+3 = (x-2)^2-1 > 0$

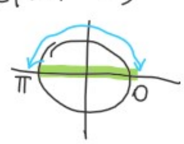
$\lambda = 1$, $x-2 = \frac{1}{\cos t}$, $dx = -\frac{1}{\cos^2 t}\cdot(-\sin t)\,dt = \frac{\sin t}{\cos^2 t}\,dt$

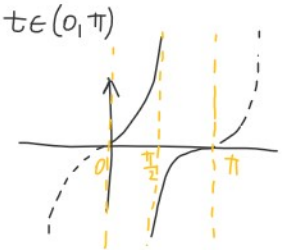
$\sqrt{(x-2)^2-1} = \sqrt{\frac{1}{\cos^2 t}-1} = \sqrt{\frac{1-\cos^2 t}{\cos^2 t}} = |tg t|$

$(x-2)^2-1>0$

$\frac{1}{\cos^2 t} > 1 \Rightarrow \cos^2 t < 1$ ✓

$|x-2|>1 \Rightarrow \quad \cos t \in (-1,1) \Rightarrow t \in (0,\pi)$





1° $t \in (0, \frac{\pi}{2})$, $tg t > 0$, $x-2>1$, $x>3$

$$I = \int\left(\frac{1}{\cos^2 t} + \frac{1}{\cos t}\right)dt$$

2° $t \in (\frac{\pi}{2}, \pi)$, $tg t < 0$, $x-2<-1$, $x<1$

$$I = -\int\left(\frac{1}{\cos^2 t} + \frac{1}{\cos t}\right)dt$$

$\int \frac{1}{\cos^2 t}\,dt = tg t + C$ ✓

$\int \frac{dt}{\ } = \int \cos t\,dt \qquad \int du$

$$\int \frac{dt}{\cos t} = \int \frac{\cos t\,dt}{1-\sin^2 t} \underset{\substack{\sin t = u \\ du = \cos t\,dt}}{=} \int \frac{du}{1-u^2} = \dots$$

$$\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right) dx$$

смена: $t = \sqrt[n]{\frac{ax+b}{cx+d}}$

пр. $\underbrace{\int \sqrt{\frac{1+x}{1-x}}\,dx}_{\text{домаћи}}$, $\int \frac{\sqrt[3]{\frac{x}{1+x}}}{\sqrt{\frac{x}{1+x}}+1}\,dx$

$\sqrt[3]{\ }, \sqrt[2]{\ } \rightsquigarrow \sqrt[6]{\ }$

$\sqrt[3]{\ } = (\sqrt[6]{\ })^2$, $\sqrt{\ } = (\sqrt[6]{\ })^3$

(3) $$\int \frac{\sqrt[3]{\frac{x}{1+x}}}{\sqrt{\frac{x}{1+x}}+1}\,dx = \int \frac{t^2}{1+t^3} \cdot \frac{6t^5}{(1-t^6)^2}\,dt = 6\int \frac{t^7\,dt}{(1+t)^3 \cdot (1-t+t^2)^3 (1-t)^2 (1+t+t^2)^2} = \dots$$ (не радити!!)

$$1-t^6 = \underbrace{(1+t)(1-t+t^2)}_{1+t^3}(1-t)(1+t+t^2)$$

$$\sqrt[6]{\frac{x}{1+x}} = t \Rightarrow \sqrt[3]{\frac{x}{1+x}} = t^2$$

$$\sqrt{\frac{x}{1+x}} = t^3$$

$$\frac{x}{1+x} = t^6$$

$$x = t^6 + x\cdot t^6$$

$$x(1-t^6) = t^6$$

$$x = \frac{t^6}{1-t^6} \Rightarrow dx = \frac{6t^5(1-t^6) - t^6(-6t^5)}{(1-t^6)^2}\,dt = \frac{6t^5}{(1-t^6)^2}\,dt$$

(4) $$\int \frac{\sqrt{x}}{1+\sqrt[4]{x}}\,dx = \int \frac{t^2}{1+t} \cdot 4t^3\,dt = 4\cdot\int \frac{t^5\,dt}{1+t} = 4\cdot\int \left(t^4 - t^3 + t^2 - t + 1 - \frac{1}{1+t}\right) dt = \frac{4}{5}t^5 - t^4 + \frac{4}{3}t^3 - 2t^2 + 4t - 4\log|1+t| + C$$

$t = x^{1/4}$

$$\sqrt[4]{x} = t \Rightarrow \sqrt{x} = t^2$$

$$x = t^4 \Rightarrow dx = 4t^3\,dt$$

$$t^5 + 1 = (t+1)(t^4 - t^3 + t^2 - t + 1)$$

Чебишевљев интеграл $\int x^p (a+bx^q)^r\,dx$, $p, q, r \in \mathbb{Q}\setminus\{0\}$

$$\int x^p \cdot (a+bx^q)^r\,dx$$

1° $r \in \mathbb{Z}$, $p = \frac{p_1}{p_2}$, $q = \frac{q_1}{q_2}$, $k = \text{НЗС}(p_2, q_2)$

$1^\circ\ r\in\mathbb{Z}$, $p=\frac{p_1}{p_2}$, $q=\frac{q_1}{q_2}$, $k=\text{НЗС}(p_2,q_2)$

смена: $\sqrt[k]{x}=t$

Ако $r\notin\mathbb{Z}$:

$$y=x^q \Rightarrow x=y^{1/q}$$
$$dx=\frac{1}{q}\cdot y^{\frac{1}{q}-1}dy$$

$$\frac{p}{q}+\frac{1}{q}-1=m$$

$$\int x^p(a+bx^q)^r dx=\int y^{p/q}\cdot(a+by)^r\cdot\frac{1}{q}\cdot y^{\frac{1-q}{q}}dy=\frac{1}{q}\int(a+by)^r\cdot y^m dy$$

$2^\circ\ m\in\mathbb{Z}$, тј. $\frac{p+1}{q}\in\mathbb{Z}$

$r=\frac{r_1}{r_2}$, смена: $\sqrt[r_2]{a+by}=t$

$3^\circ\ m\notin\mathbb{Z}$, $r+m\in\mathbb{Z}$

$$\frac{1}{q}\int(a+by)^r\cdot y^m dy=\frac{1}{q}\int\left(\frac{a+by}{y}\right)^r\cdot y^{m+r}dy$$

$r=\frac{r_1}{r_2}$, смена: $\sqrt[r_2]{\frac{a+by}{y}}=t$.

5) $\int x^2\cdot\sqrt[3]{2-3x}\,dx$

$r\in\mathbb{Z}$, $\frac{1}{3}\in\mathbb{Z}$? ×

$x^p\cdot(a+bx^q)^r$

$p=2$
$a=2$
$b=-3$
$q=1$
$r=\frac{1}{3}$

$\frac{p+1}{q}\in\mathbb{Z}$, $\frac{2+1}{1}=3\in\mathbb{Z}$ ✓

$y=x^q$ овде нема смисла (јер $y=x$) → прескачемо

$r=\frac{r_1}{r_2}=\frac{1}{3}$, $r_2=3$

$$\sqrt[3]{2-3x}=t$$
$$2-3x=t^3\ /d \Rightarrow x=\frac{2-t^3}{3}$$
$$-3dx=3t^2dt \Rightarrow dx=-t^2dt$$

$$\int x^2\cdot\sqrt[3]{2-3x}\,dx=\int\left(\frac{2-t^3}{3}\right)^2\cdot t\cdot(-t^2)dt=-\frac{1}{9}\int(2-t^3)^2\cdot t^3dt=-\frac{1}{9}\int(4-4t^3+t^6)\cdot t^3dt=\ldots$$

штимали

6) $\int\frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}dx=I$

⑥ $\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,dx = I$

$x^p(a+bx^q)^r$ $\quad \left\{ \begin{array}{l} p=-\frac{1}{2}, \ a=1, b=1 \\ q=\frac{1}{4} \\ r=\frac{1}{3} \end{array} \right.$

$x^{-1/2}\cdot\left(1+x^{1/4}\right)^{1/3}$

$r\in\mathbb{Z}, \ \frac{1}{3}\notin\mathbb{Z} \ \times$

$\frac{p+1}{q}\in\mathbb{Z}, \ \frac{-\frac{1}{2}+1}{1/4} = \frac{1/2}{1/4} = 2\in\mathbb{Z} \ \checkmark$

$x^{1/4}=y$

$x=y^4$

$dx=4y^3dy$

$I=\int y^{-2}\cdot(1+y)^{1/3}\cdot 4y^3dy = 4\int y(1+y)^{1/3}dy$

$(1+y)^{1/3}=t$

$1+y=t^3 \Rightarrow dy=3t^2dt$

$I=4\int(t^3-1)\cdot t\cdot 3t^2dt = \ldots$

⑦ $\int\frac{dx}{\sqrt[4]{1+x^4}} = \int(1+x^4)^{-1/4}dx = I$

$p=0$

$q=4$

$r=-1/4$

$r=-1/4\notin\mathbb{Z}$

$\frac{p+1}{q}=\frac{1}{4}\notin\mathbb{Z}$

$r+\frac{p+1}{q} = -1/4+1/4=0\in\mathbb{Z} \ \checkmark$

$x^4=y$

$x=y^{1/4}$

$dx=1/4\cdot y^{-3/4}dy$

$I=\int(1+y)^{-1/4}\cdot\frac{1}{4}y^{-3/4}dy = \frac{1}{4}\int\frac{(1+y)^{-1/4}}{y^{-1/4}}\cdot y^{-1}dy =$

$:y^{-1/4} \quad y^{-1/4}$

$=\frac{1}{4}\int\left(\frac{1+y}{y}\right)^{-1/4}\cdot y^{-1}dy = \cancel{\frac{1}{4}}\int \cancel{t}\cdot\frac{\cancel{1+t^4}}{\cancel{t^4}}\cdot\frac{\cancel{4}t^{\cancel{3}}}{(1+t^4)^{\cancel{2}}}dt =$

$=\int\frac{dt}{1+t^4} = \ldots$

$\left(\frac{1+y}{y}\right)^{-1/4}=t$

$\frac{1+y}{y}=t^{-4}$

$1+y=yt^{-4}$

$y(1-t^{-4})=-1$

$y=\frac{1}{\ldots}=\frac{t^4}{1+t^4}$

$t=\left(\frac{1+y}{y}\right)^{-1/4}=\left(\frac{1+x^4}{x^4}\right)^{-1/4}$

$$y = \frac{1}{t^{-4}+1} = \frac{t^4}{1+t^4}$$

$$dy = \frac{4t^3(1+t^4) - t^4 \cdot 4t^3}{(1+t^4)^2}\,dt = \frac{4t^3}{(1+t^4)^2}\,dt$$

Разни задаци

(1) $$\int \frac{dx}{1+e^{x/2}+e^{x/3}+e^{x/6}} = \left| \begin{array}{l} e^x = t \\ e^x dx = dt \end{array} \right| = \int \frac{dt}{t \cdot \left(1+\sqrt{t}+\sqrt[3]{t}+\sqrt[6]{t}\right)} = \left| \begin{array}{l} \sqrt[6]{t} = u \\ t = u^6 \\ dt = 6u^5 du \end{array} \right| = \int \frac{6u^5\,du}{u^6 \cdot (1+u^3+u^2+u)} =$$

$$= 6\int \frac{du}{u(u+1)(u^2+1)} = \ldots$$

$$\frac{1}{u(u+1)(u^2+1)} = \frac{A}{u} + \frac{B}{u+1} + \frac{Cu+D}{u^2+1}$$

(2) $$\int \frac{x^2+1}{x^4-x^2+1}\,dx \xrightarrow{\text{[illegible]}} \overset{\div x^2}{=} \int \frac{1+\frac{1}{x^2}}{x^2-1+\frac{1}{x^2}}\,dx = \left| \begin{array}{l} x - \frac{1}{x} = t \ /d \\ \left(1+\frac{1}{x^2}\right)dx = dt \\ t^2 = x^2 - 2 + \frac{1}{x^2} \\ \Rightarrow x^2 - 1 + \frac{1}{x^2} = t^2+1 \end{array} \right| = \int \frac{dt}{t^2+1} = \operatorname{arctg}\left(x - \frac{1}{x}\right) + C.$$

као раније...