Петак 02.04.2021.

1) $\int \frac{x\,dx}{x^3+1}$

$$\boxed{A^3+B^3=(A+B)(A^2-AB+B^2)}$$

$$x^3+1=(x+1)\underbrace{(x^2-x+1)}_{D=(-1)^2-4\cdot1\cdot1=1-4=-3<0} \rightsquigarrow \frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}=\frac{x}{x^3+1} \quad (*)$$

како брже наћи A, B, C?

$(*)\ /\cdot(x+1)$

$\Downarrow$

$$A+\underbrace{(x+1)}_{0}\frac{Bx+C}{x^2-x+1}=\frac{x}{x^2-x+1} \quad \leftarrow x=-1$$

$$A=\frac{-1}{1+1+1}=-\frac{1}{3}$$

$$x^2-x+1=0$$

$$x_{1/2}=\frac{1\pm i\sqrt{3}}{2}$$

$(*)\ /\cdot(x^2-x+1)$

$$\underbrace{(x^2-x+1)}_{0}\frac{A}{x+1}+(Bx+C)=\frac{x}{x+1} \quad \leftarrow x=\frac{1+i\sqrt{3}}{2}$$

$$B\cdot\frac{1+i\sqrt{3}}{2}+C=\frac{\frac{1+i\sqrt{3}}{2}}{\frac{1+i\sqrt{3}}{2}+1}=\frac{1+i\sqrt{3}}{3+i\sqrt{3}}=\frac{(1+i\sqrt{3})(3-i\sqrt{3})}{(3+i\sqrt{3})(3-i\sqrt{3})}=$$

$$=\frac{3+3+3\sqrt{3}i-i\sqrt{3}}{3^2+(\sqrt{3})^2}=\frac{6+2\sqrt{3}i}{12}$$

$$\underbrace{\left(\frac{B}{2}+C\right)}+i\,\underbrace{\frac{B\sqrt{3}}{2}}=\underbrace{\frac{1}{2}}+i\,\underbrace{\frac{\sqrt{3}}{6}}$$

$$\frac{B\sqrt{3}}{2}=\frac{\sqrt{3}}{6} \Rightarrow B=\frac{1}{3}$$

$$\frac{B}{2}+C=\frac{1}{2} \Rightarrow C=\frac{1}{3}$$

$$\int\frac{x\,dx}{x^3+1}=-\frac{1}{3}\int\frac{dx}{x+1}+\frac{1}{3}\int\frac{x+1}{x^2-x+1}dx=-\frac{1}{3}\log|x+1|+\frac{1}{6}\int\frac{2x-1}{x^2-x+1}dx+\frac{1}{2}\int\frac{dx}{x^2-x+1}$$

$$\frac{x+1}{x^2-x+1}=\frac{x-\frac{1}{2}+\frac{3}{2}}{x^2-x+1}=\frac{1}{2}\cdot\frac{2x-1}{x^2-x+1}+\frac{3}{2}\cdot\frac{1}{x^2-x+1}$$

$$d(x^2-x+1)=(2x-1)dx$$

$$\int\frac{2x-1}{x^2-x+1}dx=\int\frac{d(x^2-x+1)}{x^2-x+1}=\log|x^2-x+1|+C$$

$$x^2-x+1=x^2-2\cdot x\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2+\frac{3}{4}=$$

$$=\left(x-\frac{1}{2}\right)^2+\frac{3}{4}$$

$$\int\frac{dx}{x^2-x+1}=\int\frac{\frac{\sqrt{3}}{2}dt}{\frac{3}{4}t^2+\frac{3}{4}}=\frac{2}{\sqrt{3}}\operatorname{arctg}\left(\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right)+C$$

$$x-\frac{1}{2}=\frac{\sqrt{3}}{2}t \Rightarrow t=\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)$$

$x-\frac{1}{2}=\frac{\sqrt{3}}{2}t \Rightarrow t=\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)$

$dx=\frac{\sqrt{3}}{2}dt$

$$\Rightarrow \int \frac{x\,dx}{x^3+1}=\frac{1}{3}\log|x+1|+\log|x^2-x+1|+\frac{2}{\sqrt{3}}\operatorname{arctg}\left(\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right)+C.$$

② $\int \frac{dx}{x^4+1}$

$x^4+1>0 \Rightarrow$ нема $\mathbb{R}$ нула $\Rightarrow$ на квадратне

$$x^4+1=(x^2+\alpha x+\beta)(x^2+\gamma x+\delta)=x^4+\gamma x^3+\delta x^2+\alpha x^3+\alpha\gamma x^2+\alpha\delta x+\beta x^2+\beta\gamma x+\beta\delta$$

$$\begin{array}{ll} x^4: & 1=1 \\ x^3: & 0=\alpha+\gamma \\ x^2: & 0=\delta+\alpha\gamma+\beta \\ x: & 0=\alpha\delta+\beta\gamma \\ 1: & 1=\beta\delta \end{array}$$

$\gamma=-\alpha$

$0=\alpha\delta+\beta\gamma=\alpha(\delta-\beta) \Rightarrow 1^\circ\ \alpha=0 \Rightarrow \gamma=0 \Rightarrow \begin{array}{l}\beta+\delta=0\\ \beta\delta=1\end{array}$ ↯

$2^\circ\ \delta=\beta$

$1=\beta\delta=\beta^2 \Rightarrow \beta=\pm1$

$0=\delta+\alpha\gamma+\beta=2\beta-\alpha^2 \Rightarrow \alpha^2=2\beta=\pm2 \Rightarrow \alpha^2=2 \wedge \beta=1$

$(\alpha^2>0)$

$\alpha=\pm\sqrt{2} \Rightarrow \gamma=\mp\sqrt{2}$

$$x^4+1=(x^2+\sqrt{2}x+1)(x^2-\sqrt{2}x+1)$$

други начин: $x^4+1=x^4+2x^2+1-2x^2=(x^2+1)^2-(\sqrt{2}x)^2=(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$

$$\frac{1}{x^4+1}=\frac{B_1x+C_1}{x^2+\sqrt{2}x+1}+\frac{B_2x+C_2}{x^2-\sqrt{2}x+1} \quad \ddots$$

③ $\int \frac{x\,dx}{x^4+1}$

I начин: $x^4+1=(\quad)(\quad)$ $\ddots$

II начин: $$\int\frac{x\,dx}{x^4+1} \underset{\substack{t=x^2\\ dt=2x\,dx}}{=} \int\frac{\frac{dt}{2}}{t^2+1}=\frac{1}{2}\operatorname{arctg}t+C=\frac{1}{2}\operatorname{arctg}x^2+C$$

(4) $\int \frac{dx}{x^5+1}$

$$x^5+1=(x+1)(x^4-x^3+x^2-x+1)=\dots$$

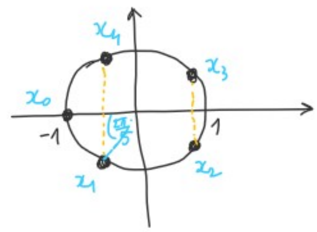
$$\boxed{A^{2k+1}+B^{2k+1}=(A+B)(A^{2k}-A^{2k-1}B+A^{2k-2}B^2-\dots+A^2B^{2k-2}-AB^{2k-1}+B^{2k})}$$

$x^5+1=0$

$x^5=-1 \to 5$ корена из -1

$\downarrow$

правилан петоугао у $\mathbb{C}$ чине нуле



$$x_k=e^{i\left(\frac{2k\pi}{5}+\pi\right)}$$

$$x_0=e^{i\pi}=-1$$

$$x_1=e^{i\frac{7\pi}{5}}=\cos\frac{7\pi}{5}+i\sin\frac{7\pi}{5}=\cos\frac{3\pi}{5}-i\sin\frac{7\pi}{5}$$

$\vdots$

$$x_4=e^{i\frac{3\pi}{5}}=\cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5}$$

$$(x-x_1)(x-x_4)=x^2-x(x_1+x_4)+x_1x_4$$
$$=x^2-2\cos\frac{3\pi}{5}x+1$$

$\overline{x_4}=x_1 \Rightarrow x_1x_4=x_4\overline{x_4}=|x_4|^2=1$

$x_2=\overline{x_3} \Rightarrow \dots \quad (x-x_2)(x-x_3)=x^2-2\cos\frac{\pi}{5}x+1$

$$(x^5+1)=(x+1)\left(x^2-\left(2\cos\frac{\pi}{5}\right)x+1\right)\left(x^2-\left(2\cos\frac{3\pi}{5}\right)x+1\right)$$

$\dots$

$\int R(\sin x,\cos x)dx$ → рац. ф-ја од sin и cos → сводимо на инт. рац. ф-ја

нпр. $\frac{\sin x}{\sin^3 x+\cos x}$

универзална смена: $t=\mathrm{tg}\frac{x}{2}$

$$\mathrm{tg}x=\frac{2\mathrm{tg}\frac{x}{2}}{1-\mathrm{tg}^2\frac{x}{2}}=\frac{2t}{1-t^2}$$

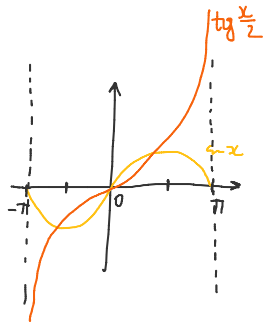
$$\mathrm{tg}^2x=\frac{\sin^2x}{\cos^2x}=\frac{\sin^2x}{1-\sin^2x} \Rightarrow \mathrm{tg}^2x-\sin^2x\cdot\mathrm{tg}^2x=\sin^2x \Rightarrow \sin^2x=\frac{\mathrm{tg}^2x}{1+\mathrm{tg}^2x}=\frac{\frac{4t^2}{(1-t^2)^2}}{1+\frac{4t^2}{(1-t^2)^2}}=$$

$\mathrm{tg}\frac{x}{2}$

$\cos^2 x \qquad 1-\sin^2 x$

$1+\mathrm{tg}^2 x \qquad 1+\frac{4t^2}{(1-t^2)^2}$

$$= \frac{4t^2}{1-2t^2+t^4+4t^2} = \frac{4t^2}{(1+t^2)^2}$$



$$\left.\begin{array}{l}\sin x > 0 \Leftrightarrow x \in (2k\pi, 2k\pi+\pi) \Leftrightarrow \mathrm{tg}\frac{x}{2} > 0 \\ \Rightarrow \mathrm{sgn}(\sin x) = \mathrm{sgn}(\mathrm{tg}\frac{x}{2}) = \mathrm{sgn}(t)\end{array}\right\} \Rightarrow \sin x = \frac{2t}{1+t^2}$$

слично: $\cos x = \frac{1-t^2}{1+t^2}$

$$\mathrm{tg}\frac{x}{2} = t \Rightarrow x = 2\,\mathrm{arctg}\, t \Rightarrow dx = \frac{2\,dt}{t^2+1}$$

$$\int R(\sin x, \cos x)\,dx \underset{t=\mathrm{tg}\frac{x}{2}}{=} \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\cdot\frac{2\,dt}{1+t^2}$$

→ интеграл рац. ф-је

(5) $$\int \frac{dx}{3+\sin x} \underset{t=\mathrm{tg}\frac{x}{2}}{=} \int \frac{\frac{2\,dt}{t^2+1}}{3+\frac{2t}{1+t^2}} = 2\int\frac{dt}{3t^2+2t+3} = \frac{2}{3}\int\frac{dt}{t^2+\frac{2}{3}t+1} = \frac{2}{3}\int\frac{dt}{\left(t+\frac{1}{3}\right)^2+\frac{8}{9}} \underset{t+\frac{1}{3}=\frac{2\sqrt{2}}{3}u}{=} \ldots$$

Трикице: 1° $R(-\sin x, \cos x) = -R(\sin x, \cos x) \rightsquigarrow t = \cos x$

2° $R(\sin x, -\cos x) = -R(\sin x, \cos x) \longrightarrow t = \sin x$

3° $R(-\sin x, -\cos x) = R(\sin x, \cos x) \longrightarrow t = \mathrm{tg}\, x$

(6) $$\int \frac{\cos^2 x}{\sin^4 x + \cos^4 x}\,dx$$

I начин: $t = \mathrm{tg}\frac{x}{2}$ → превише рачуна

II начин: $$R(-\sin x, -\cos x) = \frac{(-\cos x)^2}{(-\sin x)^4 + (-\cos x)^4} = \frac{\cos^2 x}{\sin^4 x + \cos^4 x} = R(\sin x, \cos x)$$

$$t = \mathrm{tg}\, x$$

$$\sin^2 x = \frac{\mathrm{tg}^2 x}{1+\mathrm{tg}^2 x} = \frac{t^2}{1+t^2} \ldots$$

$\cos x = \ldots$

$dx = \ldots$

$$\int \frac{\cos^2 x}{\sin^4 x+\cos^4 x}\,dx \overset{:\cos^4 x}{=} \int \frac{\frac{1}{\cos^2 x}}{\left(\frac{\sin x}{\cos x}\right)^4+1}\,dx = \int \frac{dt}{t^4+1} = \ldots$$

$t = \operatorname{tg} x$

$dt = \dfrac{dx}{\cos^2 x}$

$\int R(x, \sqrt{ax^2+bx+c})\,dx$ → сводимо на инт. рац. ф-ја

Ојлерове смене – практикуми

Тригонометријске смене – овде

$$\sqrt{ax^2+bx+c} \rightsquigarrow \sqrt{\lambda^2-x^2} \ \vee\ \sqrt{x^2+\lambda^2} \ \vee\ \sqrt{x^2-\lambda^2}$$

1) $\int R(x, \sqrt{\lambda^2-x^2})\,dx$

$x = \lambda \sin t$

(7)
$$\int \frac{x+1}{\sqrt{2x-x^2}}\,dx = \int \frac{2+t}{\sqrt{1-t^2}}\,dt = \int \frac{2+\sin u}{\sqrt{1-\sin^2 u}}\cdot \cos u\,du = \int \frac{2+\sin u}{|\cos u|}\cdot \cos u\,du = \int (2+\sin u)\,du$$

$x-1=t$

$dx = dt$

$t = \sin u$

$dt = \cos u\,du$

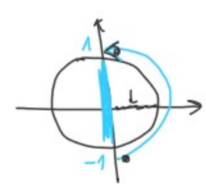
$$\sqrt{2x-x^2} = \sqrt{-(x^2-2x)} = \sqrt{-(x^2-2x+1)+1} = \sqrt{1-(x-1)^2}$$

$2x-x^2>0 \iff x(2-x)>0 \Rightarrow x\in(0,2) \Rightarrow t\in(-1,1) \Rightarrow \sin u \in(-1,1)$

$u \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\cos u > 0$

$\dfrac{\cos u}{|\cos u|} = 1$



$$\int \frac{x+1}{\sqrt{2x-x^2}}\,dx = \int (2+\sin u)\,du = 2u - \cos u + C = 2\arcsin t - \cos(\arcsin t) + C = 2\arcsin(x-1) - \cos(\arcsin(x-1)) + C.$$