$\int \sin^k x \cdot \cos^n x \, dx \quad (k,n \in \mathbb{N}_0)$

(1) (ако је неки непаран, нпр. $2 \nmid n$)

$t=\sin x$
$dt = \cos x \, dx$

$$\int \sin^2 x \cdot \cos^3 x \, dx = \int \sin^2 x \cdot \cos^2 x \cdot \underbrace{(\cos x \, dx)}_{d(\sin x)} = \int \sin^2 x \cdot (1-\sin^2 x) \cdot d(\sin x) = \int t^2 (1-t^2)\, dt = \frac{t^3}{3} - \frac{t^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

(2) (ако су оба парна)

$$\int \cos^4 x \, dx = \int \left(\tfrac{1}{2}(1+\cos 2x)\right)^2 dx = \frac{1}{4}\int (1+\cos 2x)^2 dx = \frac{1}{4}\int dx + \frac{1}{2}\int \cos 2x \, dx + \frac{1}{4}\int \cos^2 2x \, dx$$

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$

$$= \frac{x}{4} + \frac{1}{2}\cdot\frac{1}{2}\sin 2x + \frac{1}{8}\int (1+\cos 4x)\, dx =$$

$$= \frac{x}{4} + \frac{1}{4}\sin 2x + \frac{x}{8} + \frac{1}{32}\sin 4x + C$$

$\int \sin ax \cdot \cos bx \, dx$, $\int \sin ax \cdot \sin bx \, dx$, $\int \cos ax \cdot \cos bx \, dx$, $a,b \in \mathbb{R}$

$\sin ax \cdot \sin bx = \frac{1}{2}\left(\cos(a-b)x - \cos(a+b)x\right)$

$\sin ax \cdot \cos bx = \frac{1}{2}\left(\sin(a-b)x + \sin(a+b)x\right)$

$\cos ax \cdot \cos bx = \frac{1}{2}\left(\cos(a-b)x + \cos(a+b)x\right)$

нпр. $\sin(a-b)x + \sin(a+b)x =$

$= \sin ax \cdot \cos bx - \cos ax \cdot \sin bx + \sin ax \cos bx + \cos ax \sin bx$

$= 2\sin ax \cdot \cos bx$

(3) $\int \sin 2x \cdot \sin 3x \, dx = \frac{1}{2}\int (\cos x - \cos 5x)\, dx = \frac{1}{2}\sin x - \frac{1}{10}\sin 5x + C$

## Интеграција рационалних функција

$R(x) = \frac{P(x)}{Q(x)}$ полиноми

рац. ф-ја

радимо: $\int \frac{2x^2+2x+13}{(x-2)(x^4+2x^2+1)}\, dx$

пр. $R(x) = \frac{x^2+1}{(x^3-x+1)(x-1)^2}$

deg - степен полинома

1) поделити $P(x)$ и $Q(x)$ са остатком ($\deg P < \deg Q$)

код нас је ок, јер $\deg P = 2$, $\deg Q = 5$ ✓

нпр. $\int \frac{x^5+1}{x^2+1}\, dx = \int (x^3 - x)\, dx + \int \frac{x+1}{x^2+1}\, dx$

нпр. $\displaystyle\int \frac{x^5+1}{x^2+1}\,dx = \int (x^3-x)\,dx + \int \frac{x+1}{x^2+1}\,dx$

↗ остатак $\deg < \deg Q = 2$

$$x^5+1 = \underline{(x^2+1)}\cdot x^3 - x^3 + 1 = (x^2+1)x^3 - (x^2+1)\cdot x + x + 1 = (x^2+1)(x^3-x)\boxed{+x+1}$$

2) Раставимо $Q(x)$ у $\mathbb{R}[x]$, тј. на линеарне факторе $(x-\alpha)^m$ или $(ax^2+bx+c)^n$. $(b^2-4ac<0)$

код нас: $Q(x) = (x-2)(x^4+2x^2+1)$

$x^4+2x^2+1>0 \Rightarrow$ нема $\mathbb{R}$ нуле $\Rightarrow$ раставља као производ два квадратна

$$x^4+2x^2+1 = t^2+2t+1 = (t+1)^2 = (x^2+1)^2$$

$x^2=t$

$$Q(x) = (x-2)(x^2+1)^2$$

3) $R(x) = \dots$ → записати $R(x)$ као збир простих рац. ф-ја

Ако $Q(x) = \boxed{(x-\alpha_1)^{m_1}}\cdot(x-\alpha_2)^{m_2}\cdot\ldots\cdot(x-\alpha_k)^{m_k}\cdot\boxed{(a_1x^2+b_1x+c_1)^{n_1}}\cdot\ldots\cdot(a_lx^2+b_lx+c_l)^{n_l}$

$$\frac{A_{11}}{x-\alpha_1} + \frac{A_{12}}{(x-\alpha_1)^2} + \dots + \frac{A_{1m_1}}{(x-\alpha_1)^{m_1}} \qquad \frac{B_{11}x+C_{11}}{a_1x^2+b_1x+c_1} + \frac{B_{12}x+C_{12}}{(a_1x^2+b_1x+c_1)^2} + \dots + \frac{B_{1n_1}x+C_{1n_1}}{(a_1x^2+b_1x+c_1)^{n_1}}$$

просте рационалне ф-је

$$R(x) = \sum_{i=1}^{k}\sum_{j=1}^{m_i} \frac{A_{ij}}{(x-\alpha_i)^j} + \sum_{i=1}^{l}\sum_{j=1}^{n_i} \frac{B_{ij}x+C_{ij}}{(a_ix^2+b_ix+c_i)^j}$$

$$A_{ij}, B_{ij}, C_{ij} \in \mathbb{R}$$

$\boxed{T}$ Овакво растављање увек постоји и јединствено је!

код нас: $\displaystyle\int \frac{2x^2+2x+13}{(x-2)(x^4+2x^2+1)}\,dx = \int \frac{2x^2+2x+13}{(x-2)(x^2+1)^2}\,dx$

$(x-2) \rightsquigarrow \dfrac{A}{x-2}$

$(x^2+1)^2 \rightsquigarrow \dfrac{B_1x+C_1}{x^2+1} + \dfrac{B_2x+C_2}{(x^2+1)^2}$ одредити $A, B_1, C_1, B_2, C_2 = ?$

$$\frac{2x^2+2x+13}{(x-2)(x^2+1)^2} = \frac{A}{x-2} + \frac{B_1x+C_1}{x^2+1} + \frac{B_2x+C_2}{(x^2+1)^2} \quad \Big/\cdot (x-2)(x^2+1)^2$$

$x^3-2x^2+x-2$

$$\boxed{2}x^2+2x+13 = A(x^4+2x^2+1)+(B_1x+C_1)(x-2)(x^2+1)+(B_2x+C_2)(x-2)$$

$$= x^4(A+B_1)+x^3(C_1-2B_1)+x^2\boxed{(2A-2C_1+B_1+B_2)}+x(-2B_1+C_1-2B_2+C_2)+(A-2C_1-2C_2)$$

$$A+B_1=0 \Rightarrow A=-B_1$$

$$C_1-2B_1=0 \Rightarrow C_1=2B_1$$

$$2A-2C_1+B_1+B_2=2 \Rightarrow -2B_1-4B_1+B_1+B_2=2 \Rightarrow -5B_1+B_2=2$$

$$-2B_1+C_1-2B_2+C_2=2 \Rightarrow -2B_1+2B_1-2B_2+C_2=2 \Rightarrow -2B_2+C_2=2$$

$$A-2C_1-2C_2=13 \Rightarrow -B_1-4B_1-2C_2=13 \Rightarrow -5B_1-2C_2=13$$

$$B_2=2+5B_1$$

$$\left.\begin{aligned} -4-10B_1+C_2&=2 \quad /\cdot 2 \\ -5B_1-2C_2&=13 \end{aligned}\right] +$$

$$-8-25B_1=17 \Rightarrow B_1=-1 \Rightarrow B_2=-3 \Rightarrow C_2=-4$$

$$\Rightarrow A=1,\ C_1=-2$$

4) интегралити поједно просте рац. ф-је

$$\int \frac{2x^2+2x+13}{(x-2)(x^2+1)^2}\,dx = \int\frac{dx}{x-2}+\int\frac{-x-2}{x^2+1}\,dx+\int\frac{-3x-4}{(x^2+1)^2}\,dx$$

Како израчунати: $\overset{1^\circ}{\int \frac{dx}{(x-\alpha)^m}}$ и $\overset{2^\circ}{\int \frac{Bx+C}{(ax^2+bx+c)^n}\,dx}$

$1^\circ$
$$\int\frac{dx}{(x-\alpha)^m} = \Big/ \begin{matrix} x-\alpha=t \\ dx=dt \end{matrix} \Big/ = \int\frac{dt}{t^m} = \int t^{-m}\,dt = \begin{cases} \log|t|, & m=1 \\ \frac{t^{-m+1}}{-m+1}, & m\neq 1 \end{cases} = \begin{cases} \log|x-\alpha|, & m=1 \\ \frac{(x-\alpha)^{1-m}}{1-m}, & m\neq 1 \end{cases}$$

$2^\circ$
$$ax^2+bx+c = a\left(x^2+2\cdot x\cdot\frac{b}{2a}+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right)=$$

$$= a\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a}+c=$$

$$= a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}=$$

$$= a\left(\left(x+\frac{b}{2a}\right)^2+\underbrace{\overset{>0}{\frac{4ac-b^2}{4a^2}}}_{=\lambda^2}\right)$$

$$\int \frac{Bx+C}{(ax^2+bx+c)^n}dx = \int \frac{B\left(t\lambda-\frac{b}{2a}\right)+C}{\left(a\left(t^2\cdot\lambda^2+\lambda^2\right)\right)^n}\cdot\lambda dt = \int \frac{B\lambda^2 t+\left(C-\frac{Bb}{2a}\right)\lambda}{a^n\cdot\lambda^{2n}(t^2+1)^n}dt$$

$$\left[\begin{array}{l} t=\frac{1}{\lambda}\left(x+\frac{b}{2a}\right) \\ dt=\frac{1}{\lambda}dx \end{array}\right]$$

$\int \frac{t\,dt}{(t^2+1)^n}$ и $\int \frac{dt}{(t^2+1)^n}$

$$\int \frac{t\,dt}{(t^2+1)^n} = \left/ \begin{array}{l} t^2=u \\ du=2t\,dt \end{array} \right/ = \int \frac{\frac{du}{2}}{(u+1)^n} = \frac{1}{2}\int (u+1)^{-n}du = \frac{1}{2}\int v^{-n}dv = \ldots$$

$$\begin{array}{l} u+1=v \\ dv=du \end{array}$$

$$\int \frac{dt}{(t^2+1)^n} = \left/ \begin{array}{l} t=\operatorname{tg}u \\ dt=\frac{du}{\cos^2 u} \end{array} \right/ = \int \frac{\frac{du}{\cos^2 u}}{\frac{1}{\cos^{2n}u}} = \int \cos^{2n-2}u\,du = \ldots$$

$$t^2+1=\operatorname{tg}^2u+1=\frac{\sin^2u+\cos^2u}{\cos^2u}=\frac{1}{\cos^2u}$$

II начин

$$\int \frac{dt}{(t^2+1)^{n-1}} = \int \frac{(t^2+1)\,dt}{(t^2+1)^n} = \underbrace{\int \frac{t^2\,dt}{(t^2+1)^n}} + \int \frac{dt}{(t^2+1)^n} = \frac{t^2}{1-n}\frac{1}{(t^2+1)^{n-1}} - \int \frac{2t\,dt}{(1-n)(t^2+1)^{n-1}} + \int \frac{dt}{(t^2+1)^n}$$

$$I_n = \int \frac{dt}{(t^2+1)^n}$$

$$\begin{array}{ll} u=t^2 & du=2t\,dt \\ dv=\frac{dt}{(t^2+1)^n} & \Rightarrow\quad v=\frac{1}{1-n}\frac{1}{(t^2+1)^{n-1}} \end{array}$$

$$I_n = I_{n-1} - \frac{t^2}{1-n}\cdot\frac{1}{(t^2+1)^{n-1}} + \int \frac{2t\,dt}{(1-n)(t^2+1)^{n-1}} = \ldots$$

смена $t^2=u$

$$I_1 = \int \frac{dt}{t^2+1} = \operatorname{arctg}t + C$$

Код нас: $\int \frac{dx}{x-2} = \log|x-2| + C$

$$\int \frac{-x-2}{x^2+1}dx = \int \frac{-x\,dx}{x^2+1} - 2\int \frac{dx}{x^2+1} = -\frac{1}{2}\int \frac{dt}{t+1} - 2\operatorname{arctg}x = -\frac{1}{2}\log(x^2+1) - 2\operatorname{arctg}x + C$$

$$\begin{array}{l} t=x^2 \\ dt=2x\,dx \end{array}$$

$$\int \frac{-3x-4}{(x^2+1)^2}dx = -\frac{3}{2}\int \frac{2x\,dx}{(x^2+1)^2} - 4\int \frac{dx}{(x^2+1)^2} = -\frac{3}{2}\int \frac{dt}{(t+1)^2} - 4\int \frac{\frac{du}{\cos^2u}}{\frac{1}{\cos^4u}} =$$

$x^2=t$ $\quad$ $x=\operatorname{tg}u$, $du$

$$= -\frac{3}{2}(-1)\frac{1}{\quad} - 4\int \cos^2u\,du =$$

$\cos^4 u$

$x^2 = t$ $\quad$ $x = \mathrm{tg}\, u$

$dt = 2x\,dt$ $\quad$ $dx = \dfrac{du}{\cos^2 u}$

$$= -\frac{3}{2}(-1)\frac{1}{t+1} - 4\int \cos^2 u\, du =$$

$$= \frac{3}{2}\cdot\frac{1}{t+1} - 4\cdot\frac{1}{2}\int (1+\cos 2u)\, du =$$

$$= \frac{3}{2}\cdot\frac{1}{x^2+1} - 2u - \sin 2u + C =$$

$$= \frac{3}{2}\cdot\frac{1}{x^2+1} - 2\,\mathrm{arctg}\, x - \sin(2\,\mathrm{arctg}\, x) + C$$