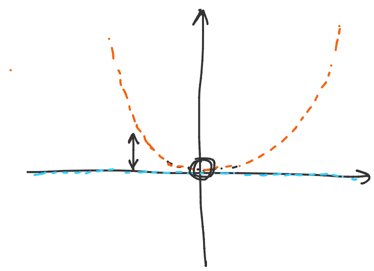
Понедељак 01.03.2021.

(1) Испитати непр. и диф. фје $f(x) = \begin{cases} x^2, & x\in\mathbb{Q} \\ 0, & x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}$



$\boxed{\text{непр.}}$

1° није непр за $x_0 \neq 0$:

ппс јесте $\Rightarrow$ Хајне $\forall$низ a_n, $a_n \to x_0 \Rightarrow f(a_n) \to f(x_0)$

$r_n \to x_0$ низ рационалних ($\mathbb{Q}$ и $\mathbb{Q}^c$ густи у $\mathbb{R}$)

$i_n \to x_0$ низ из $\mathbb{R}\setminus\mathbb{Q}$

1.1° $x_0 \in \mathbb{Q}$

$$\left.\begin{array}{l} r_n^2 = f(r_n) \to f(x_0) = x_0^2 \\ 0 = f(i_n) \to f(x_0) = x_0^2 \end{array}\right\} \Rightarrow x_0 = 0 \ ↯$$

1.2° $x_0 \notin \mathbb{Q}$

$$\left.\begin{array}{l} r_n^2 = f(r_n) \to f(x_0) = 0 \\ 0 = f(i_n) \to f(x_0) = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} r_n \to 0 \\ r_n \to x_0 \end{array}\right\} \Rightarrow x_0 = 0 \ ↯$$

2° јесте у $x_0 = 0$:

$\varepsilon > 0$

$\delta = \sqrt{\varepsilon}$

$$|x| = |x-0| < \delta = \sqrt{\varepsilon} \Rightarrow |f(x) - \underset{0}{\underset{\shortparallel}{f(0)}}| = |f(x)| = \begin{cases} x^2 \\ 0 \end{cases} \leq x^2 = |x|^2 < (\sqrt{\varepsilon})^2 = \varepsilon$$

$\boxed{\text{диф.}}$

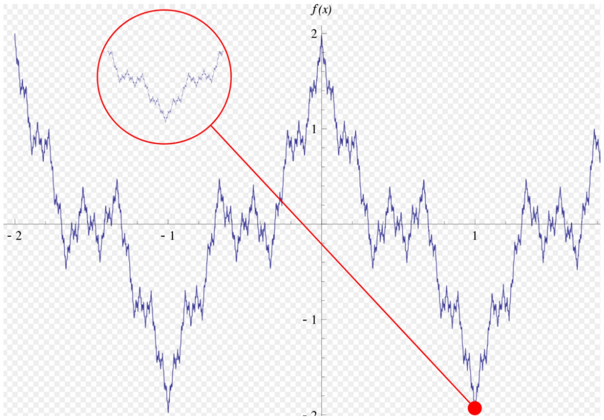
није диф за $x_0 \neq 0$!

Диф у $x_0 = 0$?

$$?\exists \lim_{h\to 0} \frac{f(0+h) - \cancelto{0}{f(0)}}{h} = \lim_{h\to 0} \frac{f(h)}{h} = 0. \quad \checkmark \qquad f'(0) = 0.$$

$$0 \leq f(x) \leq x^2 \Rightarrow 0 = \frac{0}{x} \leq \frac{f(x)}{x} \leq \frac{x^2}{x} = x \ \Big/ \lim_{x\to 0}$$

$$\lim_{x\to 0} \frac{f(x)}{x} = 0$$



Weierstrass function

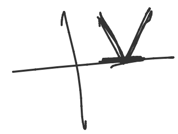
https://en.wikipedia.org/wiki/Weierstrass_function

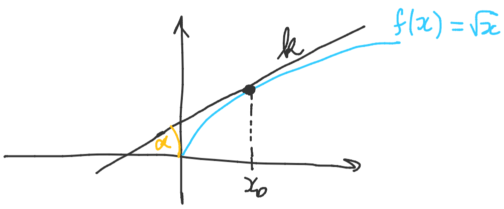
(2) Дата је крива $C = \{ (x, \sqrt{x}) \mid x \geq 0 \} \subseteq \mathbb{R}^2$

а) Наћи угао између x-осе и тангенте на C у $\left(\frac{3}{4}, \frac{\sqrt{3}}{2}\right) \in C$.

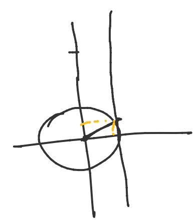
б) $(x_0, y_0) \in C$? тд. тангента сече x-осу под углом $\frac{\pi}{4}$.

в) Наћи тант. која пролази кроз $(-1,0)$.





$f'(x_0)$ = коеф. правца k тангенте у x_0 $= tg(\alpha)$



а) $x_0 = \frac{3}{4}$

$f(x_0) = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

$$f'(x_0) = (\sqrt{x})' \Big|_{x=x_0} = \left(\frac{1}{2\sqrt{x}}\right)\Big|_{x=x_0} = \frac{1}{2\cdot\sqrt{\frac{3}{4}}} = \frac{1}{\sqrt{3}} = k = tg(\alpha)$$

$\left(\frac{1}{2\sqrt{x_0}}\right)$

$$tg\left(\frac{\pi}{6}\right) = \frac{\sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}. \Big\} \alpha = \frac{\pi}{6}$$

б) $k = tg\left(\frac{\pi}{4}\right) = 1$

$f'(x_0) = 1 \Rightarrow \frac{1}{2\sqrt{x_0}} = 1 \Rightarrow x_0 = \frac{1}{4}$

$y_0 = \sqrt{x_0} = \frac{1}{2}$

$\left(\frac{1}{4}, \frac{1}{2}\right) \in C$

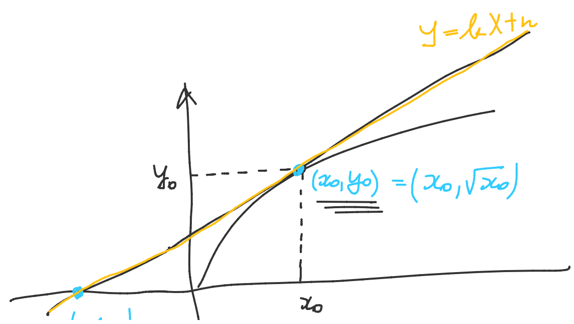
в) (x_0, y_0) ?

k ?

$\begin{matrix} X & y \\ (-1, & 0) \end{matrix} \rightarrow y = kX + n$

$\rightarrow k = \frac{1}{2\sqrt{x_0}}$

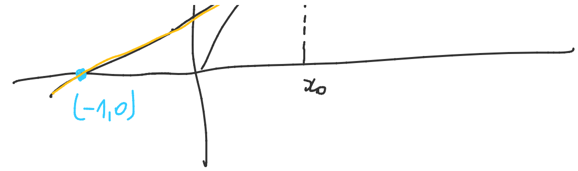
$X = -1$) $0 = -k + n \Rightarrow k = n.$



$(-1,0) \to y = kx + n$

$$\left.\begin{array}{l} x=-1 \\ y=0 \end{array}\right\} \; 0 = -k+n \Rightarrow \underline{k=n}$$

$2\sqrt{x_0}$



$\Downarrow$

$k = n = \dfrac{1}{2\sqrt{x_0}}$

$$\left.\begin{array}{l} x=x_0 \\ y=\sqrt{x_0} \end{array}\right\} \; \sqrt{x_0} = k\cdot x_0 + n$$

$$= \frac{1}{2\sqrt{x_0}}\cdot x_0 + \frac{1}{2\sqrt{x_0}} = \frac{\sqrt{x_0}}{2} + \frac{1}{2\sqrt{x_0}}$$

$$\frac{\sqrt{x_0}}{\cancel{2}} = \frac{1}{\cancel{2}\sqrt{x_0}} \Rightarrow x_0 = 1 \Rightarrow y_0 = 1 \Rightarrow k = n = \frac{1}{2}$$

$$\boxed{y = \frac{1}{2}x + \frac{1}{2}}$$

$(1,1) \in C$

③ Израчунати суму $S_n(x) = 1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \dots + n^2x^{n-1}$

$\underline{x=1:}$

$S_n(1) = 1^2 + 2^2 + \dots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

$$1 + x + x^2 + \dots + x^n = \frac{1-x^{n+1}}{1-x} \;/'\quad, \underline{x \neq 1}$$

$$\to 0 + 1 + 2x + 3x^2 + \dots + nx^{n-1} = \frac{-(n+1)x^n\cdot(1-x) + (1-x^{n+1})}{(1-x)^2} \;/\cdot x$$

$$x + 2x^2 + 3x^3 + \dots + nx^n = x\cdot(\,-\,-\,) \;/'$$

$$S_n(x) = 1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \dots + n^2x^{n-1} = \big(x\cdot(\,-\,-\,)\big)'$$

(ручно)

④ (Извод инверзне функције, $f(x) = y$, $\exists f^{-1}$, $(f^{-1}(y))' = \dfrac{1}{f'(f^{-1}(y))} = \dfrac{1}{f'(x)}$)

а) $(\arccos x)' = -\dfrac{1}{\sqrt{1-x^2}}$, $x \in (-1,1)$

б) $(\operatorname{arctg} x)' = \dfrac{1}{1+x^2}$

в) $(\operatorname{arcctg} x)' = -\dfrac{1}{1+x^2}$

доказати

$\left(\cos^2 x = \dfrac{1}{1+\operatorname{tg}^2 x}\right)$

а) $f(x) = \cos x = y$

$f^{-1}(y) = \arccos y$

$(f^{-1}(y))' = \dfrac{1}{f'(x)}$

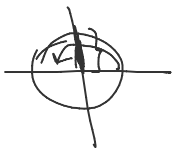
$f^{-1}(y) = \arccos y$ $\qquad (f^{-1}(y))' = \frac{1}{f'(x)}$

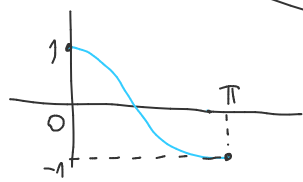
$$(\arccos y)' = \frac{1}{-\sin x} = -\frac{1}{\sqrt{1-\cos^2 x}} = -\frac{1}{\sqrt{1-y^2}}$$

$\left.\begin{array}{l} y \in (-1,1) \\ x \in (0,\pi) \end{array}\right\} \Rightarrow \sin x > 0$

$\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x \Rightarrow \sin x = \sqrt{1-\cos^2 x}$





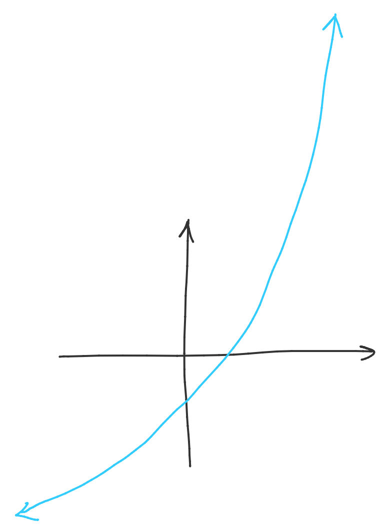
⑤ Доказати да $f(x) = x^3 + 3x$ има инверзну фју f^{-1} и одредити $(f^{-1})'(0)$.

$f^{-1} = g \quad , g'(0) = ?$

$y = x^3 + 3x \Rightarrow \ldots \quad x = \ldots$ ТЕШКО!

↑ нешто од y

$f^{-1} = g$ не можемо да добијемо аналитички.



$f'(x) = 3x^2 + 3 > 0 \Rightarrow f$ је строго растућа

$\Rightarrow f$ је 1-1

$\left.\begin{array}{l} \lim\limits_{x\to\infty} f(x) = \infty \\ \lim\limits_{x\to-\infty} f(x) = -\infty \end{array}\right\} f(\mathbb{R}) = \mathbb{R} \Rightarrow f$ је на

$\}$ f бијекција $\Rightarrow \exists f^{-1} = g$

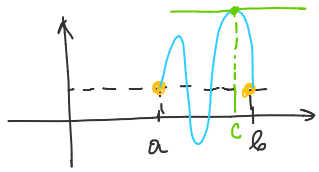
$$g'(y) = \frac{1}{f'(x)} = \frac{1}{3x^2+3} \Rightarrow g'(0) = \frac{1}{3\cdot 0^2 + 3} = \frac{1}{3}.$$

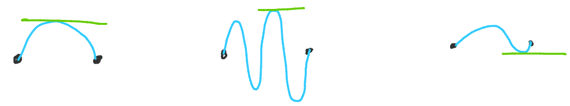
$y = 0 \Leftrightarrow x = ?$ $\qquad f(0) = 0 \Rightarrow f^{-1}(0) = 0$

$g(0) = 0$

T (Ролова) $f : [a,b] \to \mathbb{R}$ непр. $(a \neq b)$ и f диф. на (a,b) и $f(a) = f(b) \Rightarrow \exists c \in (a,b), f'(c) = 0.$

[1] (Ролова) $f:[a,b]\to\mathbb{R}$ непр. и f диф. на (a,b) и $f(a)=f(b) \Rightarrow \exists c\in(a,b),\ f'(c)=0.$
$(a\neq b)$





„класе C^1"

$f\in C^1(D)$ ако
је f (непр. и) диф. на D
и f' непр. на D

(4) са прошлог часа:
јесте диф, није C^1

(1) $f\in C^1(\mathbb{R})$ и $f(x_0)=0$. Док. $(\exists c\in\mathbb{R})$ $\boxed{cf'(c)+f(c)=0}$

$g'(c)=0$

(C^1-непрекидно диференцијабилна)

Рол на f? $f(x_0)=0$ ×

Која ф-ја?

$$g(x)=x\cdot f(x)\in C^1(\mathbb{R})$$

$$g'(x)=(xf(x))'=(x)'\cdot f(x)+x\cdot f'(x)=xf'(x)+f(x)$$

$$g(x_0)=x_0\cdot\underbrace{f(x_0)}_{0}=0$$

$$g(0)=0\cdot f(0)=0$$

1° $x_0\neq 0 \overset{\text{Рол}}{\Rightarrow} (\exists c\in\mathbb{R})\ g'(c)=0 \Rightarrow cf'(c)+f(c)=0$

$\left[\begin{array}{l} c\in(0,x_0) \\ c\in(x_0,0)\end{array}\right]$

2° $x_0=0$
$f(0)=0$

$$g'(0)=\left(xf'(x)+f(x)\right)\Big|_{x=0}=\underbrace{0\cdot f'(0)}_{0}+\underbrace{f(0)}_{0}=0.$$

(2) Доказати да ј-на $a_1\cos x+a_2\cos 2x+\dots+a_n\cos nx=0$ има решење $x\in(0,\pi)$.
$(a_1,\dots,a_n\in\mathbb{R})$

$$f'(x)=a_1\cos x+\dots+a_n\cos nx \quad [/\textstyle\int]$$

$$a_k\cos kx=\left(\frac{a_k}{k}\sin kx\right)'=\frac{a_k}{k}\cdot(\cos kx\cdot k) \Rightarrow f(x)=\frac{a_1}{1}\sin x+\frac{a_2}{2}\sin 2x+\frac{a_3}{3}\sin 3x+\dots+\frac{a_n}{n}\sin nx$$

f је C^1 ✓

$\sin k\pi=0$

$f(0)=0$)

f је C^1 ✓

$\left.\begin{array}{l} f(0)=0 \\ f(\pi)=0 \end{array}\right\} \leftarrow \sin k\pi = 0$

Рол $\Rightarrow$ $(\exists c\in(0,\pi))$ $f'(c)=0$ ✓

(3) Нека је $f\in C^1(\mathbb{R})$ и $f(0)=f(1)=0$. Док. $(\forall\lambda\in\mathbb{R})(\exists c\in(0,1))$ $f'(c)=\lambda\cdot f(c)$

домаћи! Хинт: $g(x)=e^{-\lambda x}\cdot f(x)$

(4) $f:\mathbb{R}\to\mathbb{R}$ је 3 пута диференцијабилна и $f(0)=f'(0)=f(1)=f'(1)=0$ $\Rightarrow$ $(\exists c\in(0,1))$ $f'''(c)=0$.

деф. f је 2 пута диф. ако је f диф. и f' диф.

$f''(x)=(f')'(x)$

$\vdots$

f је k пута диф. ако је $k-1$ пута диф. и $f^{(k-1)}$ је диф.

$f', f'', f''', f^{(4)}, f^{(5)}, \ldots$

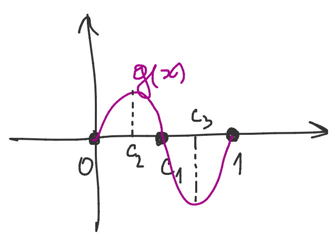
$f(0)=f(1)=0 \overset{\text{Рол}}{\Rightarrow} (\exists c_1\in(0,1))$ $f'(c_1)=0$

$g(x)=f'(x)$

$g(0)=f'(0)=0$

$g(1)=f'(1)=0$

$g(c_1)=f'(c_1)=0$



Рол на g на $[0,c_1]$: $g(0)=g(c_1)=0 \Rightarrow (\exists c_2\in(0,c_1))$ $g'(c_2)=0$

Рол на g на $[c_1,1]$: $g(c_1)=g(1)=0 \Rightarrow (\exists c_3\in(c_1,1))$ $g'(c_3)=0$

$h(x)=g'(x)$ $\qquad 0<c_2<c_1<c_3<1$

$\left.\begin{array}{l} h(c_2)=g'(c_2)=0 \\ h(c_3)=g'(c_3)=0 \end{array}\right\}$ Рол на h на $[c_2,c_3] \Rightarrow (\exists c\in(c_2,c_3))$ $h'(c)=0$.

$0=h'(c)=g''(c)=f'''(c)$.