$\boxed{\text{Линеарне ДЈ вишег реда}}$

$$y^{(n)}+a_{n-1}y^{(n-1)}+\dots+a_1y'+a_0y=g(x)$$

<u>Хомогене:</u> $(g(x)\equiv 0)$

карактеристична једначина: $\lambda^n+a_{n-1}\lambda^{n-1}+\dots+a_1\lambda+a_0=0$

Имамо n решења.

1° реално λ_0, вишеструкости m $\to e^{\lambda_0 x}, xe^{\lambda_0 x}, x^2e^{\lambda_0 x}, \dots, x^{m-1}e^{\lambda_0 x}$

(m укупно)

2° комплексно $a+ib$, виш. m

$\hookrightarrow e^{ax}\cos bx, e^{ax}\sin bx, xe^{ax}\cos bx, xe^{ax}\sin bx, x^2e^{ax}\cos x, x^2e^{ax}\sin bx$

$\dots, x^{m-1}e^{ax}\cos bx, x^{m-1}e^{ax}\sin bx$

($2m$ укупно)

ОР = линеарна комбинација придружених

1. а) $y'''-13y'-12y=0$

$\lambda^3-13\lambda-12=0 \Rightarrow \lambda_1=-3, \lambda_2=-1, \lambda_3=4$

$\downarrow \quad e^{-3x} \qquad e^{-x} \qquad e^{4x}$

$y(x)=c_1e^{-3x}+c_2e^{-x}+c_3e^{4x}$, $c_1,c_2,c_3\in\mathbb{R}$

б) $y'''-7y''+16y'-12y=0$

$\lambda^3-7\lambda^2+16\lambda-12=0 \to \lambda_1=3, \lambda_2=\lambda_3=2$

$\downarrow \quad e^{3x} \qquad e^{2x}, xe^{2x}$

в) $y'''-3y''+9y'+13y=0$

$\lambda^3-3\lambda^2+9\lambda+13=0 \to \lambda_1=-1, \lambda_{2/3}=2\pm 3i$

$\downarrow \quad e^{-x} \qquad e^{2x}\cos 3x, e^{2x}\sin 3x$

г) $y^{(6)}-4y^{(5)}+8y^{(4)}-8y^{(3)}+4y''=0$

$\lambda^6-4\lambda^5+8\lambda^4-8\lambda^3+4\lambda^2=0$

$\lambda^2(\lambda^2-2\lambda+2)=0 \to \lambda_1=\lambda_2=0, \lambda_{3/4}=\lambda_{5/6}=1\pm i$

$\downarrow \quad 1, x \qquad e^x\cos x, e^x\sin x, xe^x\cos x, xe^x\sin x$

ОР: $y(x)=c_1+c_2x+c_3e^x\cos x+c_4e^x\sin x+c_5xe^x\cos x+c_6xe^x\sin x$

$c_1,c_2,c_3,c_4,c_5,c_6\in\mathbb{R}$

② Решити Кошијев проблем: $y''' + y'' = 0$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = 1$

$\lambda^3 + \lambda^2 = \lambda^2(\lambda+1) = 0$

ОР: $y(x) = c_1 + c_2 x + c_3 e^{-x}$, $c_1, c_2, c_3 = ?$

$y'(x) = c_2 - c_3 e^{-x}$

$y''(x) = c_3 e^{-x}$

систем: $\left.\begin{aligned} c_1 + c_3 &= 1 \\ c_2 - c_3 &= 0 \\ c_3 &= 1 \end{aligned}\right\}$ $\begin{aligned} c_1 &= 0 \\ c_2 &= 1 \\ c_3 &= 1 \end{aligned}$

КР: $y_K(x) = x + e^{-x}$

Нехомогене: ако је $g(x) = e^{\alpha x}(P_n(x) \cdot \cos\beta x + Q_m(x) \cdot \sin\beta x)$

s-вишеструкост $\alpha + i\beta$ као корена карактер. јне

партикуларно $\longrightarrow$ $y_P(x) = x^s \cdot e^{\alpha x}(\overline{P_k}(x) \cdot \cos\beta x + \overline{Q_k} \cdot \sin\beta x)$, $k = \max\{n, m\}$

ОР: $y(x) = y_H(x) + y_P(x)$

③ а) $y''' - y'' + y' - y = x^2 + x$

$\lambda_1 = 1$, $\lambda_{2/3} = \pm i$ $\rightsquigarrow$ $y_H(x) = c_1 e^x + c_2 \cos x + c_3 \sin x$

$\left.\begin{aligned} &\alpha = 0 \\ &\beta = 0 \quad (\cos = 1, \sin = 0) \\ &\alpha + i\beta = 0 \Rightarrow s = 0 \end{aligned}\right\}$ $g(x) = e^{0x}(P_n(x) \cdot 1 + Q_m(x) \cdot 0)$

$= P_n(x) = x^2 + x$

$\left.\begin{aligned} n &= 2 \\ m &= 0 \end{aligned}\right\} \Rightarrow k = 2$

$y_P(x) = x^0 \cdot e^{0x}(\overline{P_2}(x) \cdot 1 + \overline{Q_2} \cdot 0) = \overline{P_2}(x) = ax^2 + bx + c$

$y_P''' - y_P'' + y_P' - y_P = x^2 + x \Rightarrow -ax^2 + (2a - b)x + (-2a + b - c) = 0$

$\vdots\ a = -1$, $b = -3$, $c = -1$

ОР: $y(x) = c_1 e^x + c_2 \cos x + c_3 \sin x - x^2 - 3x - 1$

б) $y''' - y'' + y' - y = \underbrace{\cos x}_{g_1(x)} + \underbrace{2e^x}_{g_2(x)}$ $y(x) = y_H(x) + y_{P_1}(x) + y_{P_2}(x)$

1° $g_1(x) = \cos x$

$\left.\begin{aligned} &\alpha = 0, \beta = 1 \\ &\alpha + i\beta = i \longrightarrow s = 1 \\ &\left.\begin{aligned} n &= 0 \\ m &= 0 \end{aligned}\right\} k = 0 \end{aligned}\right\}$ $y_{P_1}(x) = x(c_1 \cos x + c_2 \sin x)$

$\vdots\ c_1 = c_2 = -\frac{1}{4}$

$2^\circ\ g_2(x)=2e^x$

$$\left.\begin{array}{l}\left.\begin{array}{l}\alpha=1\\ \beta=0\end{array}\right\}\ s=1\\ \left.\begin{array}{l}n=0\\ m=0\end{array}\right\}\ k=0\end{array}\right\}\ y_{P_2}(x)=x e^x\cdot c \Rightarrow \dots c=1$$

$$\text{OP}:\ y(x)=c_1e^x+c_2\cos x+c_3\sin x-\frac{x}{4}(\cos x+\sin x)+xe^x$$

в) $y''-y=\underbrace{\sin^2 x}_{\shortparallel}$

$$\frac{1}{2}(1-\cos 2x)=\frac{1}{2}-\frac{\cos 2x}{2}\quad \vdots$$

доматп! $\longrightarrow$ решење: $y(x)=c_1e^x+c_2e^{-x}-\frac{1}{2}+\frac{1}{10}\cos 2x$

г) $y''-4y'+5y=(\sin x+2\cos x)e^{2x}$

$$\left.\begin{array}{l}\alpha=2\\ \beta=1\end{array}\right\}\ 2+i\Rightarrow s=1\ ,\ \left.\begin{array}{l}n=0\\ m=0\end{array}\right\}\ k=0\ ,\ y_P(x)=xe^{2x}(c_1\cos x+c_2\sin x)$$

$\vdots\ c_1=-\frac{1}{2},\ c_2=1$

4 $y''+4y=\underbrace{2\operatorname{tg}x}$ $\longrightarrow$ НИЈЕ у оном облику!

$z=y' \Rightarrow z'+4y=2\operatorname{tg}x$

$$\begin{aligned} y'&=z\\ z'&=-4y+2\operatorname{tg}x\end{aligned}\qquad g(x)=\begin{bmatrix}0\\ 2\operatorname{tg}x\end{bmatrix}$$

$\Phi(x),\ \int\Phi^{-1}(x)\cdot g(x)\,dx,\ \dots$

$$y(x)=c_1\cos 2x+c_2\sin 2x-x\cos 2x+\sin x\cdot\ln|\cos x|\ ,\quad c_1,c_2\in\mathbb{R}$$

5 $y''=y+z$ $\longrightarrow$ $z=y''-y,\ z'=y'''-y',\ z''=y^{(4)}-y''$

$z''-7z'+18y''-24y'+6y=0$

$$y^{(4)}-y''-7(y'''-y')+18y''-24y'+6y=0$$

$$\Downarrow$$

$$y^{(4)}-7y'''+17y''-17y'+6y=0$$

$\vdots$

Ојлерова једначина: $(ax+b)^n y^{(n)} + p_1(ax+b)^{n-1} y^{(n-1)} + \dots + p_{n-1}(ax+b)y' + p_n y = 0$

$a, b, p_1, \dots, p_n \in \mathbb{R}$

тјд: $a=1, b=0 \rightarrow$ најчешћи облик

смена: $t = \ln|ax+b|$ (тј. $t=\ln|x|$ за $a=1, b=0$)

$$y'_x = y'_t \cdot t'_x$$

$$y''_{xx} = (y'_t \cdot t'_x)'_x = (y'_t)'_x \cdot t'_x + y'_t \cdot t''_{xx} = y''_{tt} \cdot (t'_x)^2 + y'_t \cdot t''_{xx}$$

$$\Rightarrow y'_x = y'_t \cdot \frac{1}{ax+b}$$

$$y''_{xx} = y''_{tt} \cdot \frac{1}{(ax+b)^2} + y'_t \cdot \left(-\frac{a}{(ax+b)^2}\right) = \frac{1}{(ax+b)^2}\left(y''_{tt} - a y'_t\right)$$

своди се на ЛДЈКК!

6) $x^2 y'' - x y' + 4y = \cos(\ln x) + x \cdot \sin(\ln x)$

$x > 0$ (због $\ln x$)

$t = \ln|x| = \ln x$

$\Rightarrow x = e^t$

$$x \cdot y'_x = y'_t$$

$$x^2 \cdot y''_{xx} = y''_{tt} - y'_t$$

$$\Rightarrow y''_{tt} - 2y'_t + 4y = \underbrace{\cos t}_{g_1(t)} + \underbrace{e^t \cdot \sin t}_{g_2(t)}$$

$$y_{p_1}(t) = \frac{3}{13}\cos t - \frac{2}{13}\sin t$$

$$y_H(t) = c_1 e^t \cos(t\sqrt{3}) + c_2 e^t \sin(t\sqrt{3})$$

$$y_{p_2}(t) = \frac{e^t}{2}\sin t$$

ОР: $y(x) = c_1 x \cos(\sqrt{3}\ln x) + c_2 x \sin(\sqrt{3}\ln x) + \frac{3}{13}\cos(\ln x) - \frac{2}{13}\sin(\ln x) + \frac{x}{2}\sin(\ln x)$

$c_1, c_2 \in \mathbb{R}$