⑤ Решити систем
$$y_1' = 2y_1 + y_3$$
$$y_2' = y_2 + y_4$$
$$y_3' = 2y_3 + y_4$$
$$y_4' = -y_2 + y_4$$

$$A = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad ; \quad \lambda_1 = \lambda_2 = 2 \quad , \lambda_{3/4} = 1 + i$$

$\lambda_1 = 2: \quad (A - \lambda E)\gamma_1 = 0 \rightsquigarrow \gamma_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\quad (A - \lambda_1 E)^2 \gamma_2 = 0 \rightsquigarrow \gamma_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_3 = 1 + i: \quad \gamma_3 = \begin{bmatrix} 1 \\ -2 \\ i-1 \\ -2i \end{bmatrix}$

$$D = \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 1 & 1 \\ & & -1 & 1 \end{bmatrix} \begin{array}{l} \\ \end{array}$$
$$\underbrace{\quad}_{\lambda_1} \underbrace{\quad}_{\lambda_2} \underbrace{\quad\quad}_{\lambda_{3/4}}$$

$$S = \begin{array}{cccc} & & \mathrm{Re} & \mathrm{Im} \\ \end{array}$$
$$S = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \longrightarrow S^{-1} = \frac{1}{2} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$Q = SDS^{-1} = \frac{1}{2} \begin{bmatrix} 4 & 1 & 0 & -1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 4 & 2 \\ 0 & -2 & 0 & 2 \end{bmatrix} , \quad N = A - Q = \frac{1}{2} \begin{bmatrix} 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , N^2 = 0$$

$$\text{OP}: \; y(x) = e^{Ax} \cdot c = e^{Qx} \cdot e^{Nx} \cdot c = S e^{Dx} S^{-1} \cdot (E + Nx) \cdot c$$

$$e^{Dx} = \begin{bmatrix} e^{2x} & & & \\ & e^{2x} & & \\ & & e^x \cos x & e^x \sin x \\ & & -e^x \sin x & e^x \cos x \end{bmatrix}$$

Нехомогени системи

$$y'(x) = A(x) \cdot y(x) + g(x)$$

$y(x) = \phi(x) \cdot c$ OP хомогене, $\phi(x)$ – фундаментална матрица $\Rightarrow \phi'(x) = A(x) \cdot \phi(x)$

Варијација константе: $y(x) = \phi(x) \cdot c(x)$

$$y'(x) = \phi'(x) \cdot c(x) + \phi(x) \cdot c'(x)$$

$$\underline{\phi'(x)} \cdot c(x) + \phi(x) \cdot c'(x) = A(x) \cdot y(x) + g(x) \Rightarrow \underline{A(x) \cdot \phi(x) \cdot c(x)} + \phi(x) \cdot c'(x) = \underline{A(x) \cdot \phi(x) \cdot c(x)} + g(x)$$

$$g(x) = \phi(x) \cdot c'(x) \Rightarrow c(x) = \int \phi^{-1}(x) \cdot g(x)\, dx + c$$

$\frac{d}{dx}$ и $\int dx$ су члан по члан!

$$\Rightarrow \quad y(x) = \phi(x) \cdot \left(c + \int \phi^{-1}(x) \cdot g(x)\, dx \right)$$

① $y_1' = y_2 + \frac{1}{\cos^2 x}$
$y_2' = -y_1 + tg\, x$ $\Bigg\}$ $g(x) = \begin{bmatrix} \frac{1}{\cos^2 x} \\ tg\, x \end{bmatrix}$; $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

хомоген: $y_1' = y_2$
$y_2' = -y_1$ $\Bigg\}$ $\Phi(x) = e^{xA} = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$

$$\Phi^{-1}(x)\cdot g(x) = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\cos^2 x} \\ tg\, x \end{bmatrix} = \begin{bmatrix} \cos x \\ \frac{\sin x}{\cos^2 x} + \sin x \end{bmatrix}$$

$$\int \Phi^{-1}(x) g(x)\, dx = \begin{bmatrix} \int \cos x\, dx \\ \int \left(\frac{\sin x}{\cos^2 x} + \sin x\right) dx \end{bmatrix} + C = \begin{bmatrix} \sin x \\ \frac{1}{\cos x} - \cos x \end{bmatrix} + C$$

$$y(x) = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \cdot \left(C + \begin{bmatrix} \sin x \\ \frac{1}{\cos x} - \cos x \end{bmatrix} \right) = \underbrace{\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \cdot C}_{\text{решење хомогене}} + \underbrace{\begin{bmatrix} tg\, x \\ 0 \end{bmatrix}}_{\text{партикуларно}}$$

$C \in \mathbb{R}^2$

② $y_1' = y_1 + y_2 - \cos x$
$y_2' = -2y_1 - y_2 + \sin x + \cos x$ $\xrightarrow{\text{домаћи}}$ $\begin{bmatrix} \cos x & \sin x \\ \cos x + \sin x & \cos x - \sin x \end{bmatrix} \cdot C + \begin{bmatrix} -x\cos x \\ x(\cos x + \sin x) \end{bmatrix}$

Смена независне промењиве

Ако имамо систем $f(x)\cdot y'(x) = Ay(x)$, $f: \mathbb{R} \to \mathbb{R}$

Хоћемо $y(x) \rightsquigarrow y(t)$

$y_t' = \underbrace{y_x' \cdot x_t'}_{y_x' \cdot f(x)}$ $\Bigg\}$ $x_t' = f(x)$, $t_x' = \frac{1}{f(x)}$

① $2\sqrt{x}\cdot y' = 2y - z$
$2\sqrt{x}\cdot z' = y + 2z$ $\xrightarrow[y(x) \longrightarrow y(t)]{\text{после смене}}$ $y_t' = 2y - z$
$z_t' = y + 2z$

$t_x' = \frac{1}{2\sqrt{x}} \rightsquigarrow t(x) = \sqrt{x} \Rightarrow y_t' = y_x' \cdot x_t' = y_x' \cdot \frac{1}{t_x'} = y_x' \cdot 2\sqrt{x}$

② $xy_1' = y_1 + y_2$
$xy_2' = y_2$
$xy_3' = -y_3$

$1^\circ\ x > 0:\ t(x) = \ln(x)$ $2^\circ\ x < 0:\ t(x) = \ln(-x)$

$\vdots$

$y_{1t}' = y_1 + y_2$
$y_{2t}' = y_2$
$y_{3t}' = -y_3$

$t_x' = \frac{1}{x} \rightsquigarrow t(x) = \ln|x|$

за $x = 0 \Rightarrow y_1(0) = y_2(0) = y_3(0) = 0$

Систем $\longleftrightarrow$ једначина вишег реда

1. $$\begin{cases} y' = py - qz \\ z' = qy + pz \end{cases} \qquad p, q \in \mathbb{R}\setminus\{0\}$$

$$z = \frac{1}{q}(py - y') \Rightarrow z' = \frac{1}{q}(py' - y'')$$

$$\frac{1}{q}(py' - y'') = qy + p\cdot\frac{1}{q}(py - y')$$

$$\Rightarrow py' - y'' = q^2 y + p^2 y - py'$$

$$\Rightarrow y'' - 2py' + (p^2 + q^2)y = 0$$

2. $$y''' - 2y'' + y = 0$$

$$y'' = y_1 \Rightarrow y_1' = y'''$$

$$y' = y_2 \Rightarrow y_2' = y'' = y_1$$

$$\Rightarrow y_1' - 2y_1 + y = 0$$

$$\Rightarrow \begin{cases} y_1' = 2y_1 + y \\ y_2' = y_1 \\ y' = y_2 \end{cases}$$

3. $$y''' = xy'' + y - z' + x + 1$$
$$z'' = y'\sin x + y - z + x^2$$

$$z_1 = z'$$
$$y_1 = y''$$
$$y_2 = y'$$

$$\begin{cases} y_1' = xy_1 + y - z_1 + x + 1 \\ y_2' = y_1 \\ y' = y_2 \\ z_1' = y_2 \sin x + y - z + x^2 \\ z' = z_1 \end{cases}$$