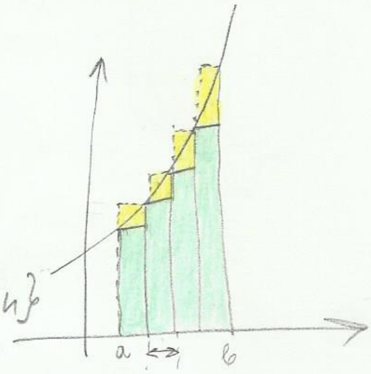
Риман–Стилтјесов интеграл

Увод: део 6.6. у скрипти ДМ

део 12 у скрипти ЈН



$$\int_a^b f(x)\,dg(x)=\lim_{n\to\infty}\sum_{k=1}^{n} f\left(a+\frac{b-a}{n}k\right)\left(g\left(a+\frac{k}{n}(b-a)\right)-g\left(a+\frac{k-1}{n}(b-a)\right)\right)$$

$$P=\left\{a,\ a+\frac{b-a}{n},\ a+\frac{b-a}{n}\cdot 2,\ \ldots,\ a+\frac{b-a}{n}\cdot n\right\}$$

f – непрекидна, g ограничене варијације на $[a,b]$

$g(x)=x \to$ своди се на Риманов интеграл

$\boxed{\mathrm{I}}$ $f\in C[a,b],\ g\in C^1[a,b] \Rightarrow$ $$\underbrace{\int_a^b f(x)\,dg(x)}_{\text{РС}}=\underbrace{\int_a^b f(x)\cdot g'(x)\,dx}_{\text{Р}}$$

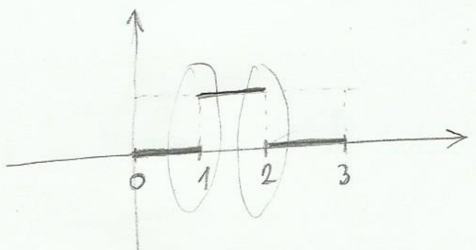
(1) $\chi_A:\mathbb{R}\to\{0,1\}$, $A\subseteq\mathbb{R}$ $\qquad \chi_A(x)=\begin{cases}0, & x\notin A\\ 1, & x\in A\end{cases}$

$f\in C[0,3]$

$$\int_0^3 f(x)\,d\chi_{[1,2]}(x)=\lim_{n\to\infty}\sum_{k=1}^{3n} f(x_k)\cdot\left(g(x_k)-g(x_{k-1})\right)=\lim_{n\to\infty}\sum_{k=1}^{3n} f\left(\tfrac{k}{n}\right)\cdot\left(g\left(\tfrac{k}{n}\right)-g\left(\tfrac{k-1}{n}\right)\right)$$

$P=\left\{0,\frac{1}{n},\frac{2}{n},\ldots,\frac{3n-1}{n},\frac{3n}{n}\right\}$, $x_k=\frac{k}{n}$

$\chi_{[1,2]}$:



$$=\lim_{n\to\infty}\left(\underbrace{f\left(\tfrac{n}{n}\right)\cdot\left(g\left(\tfrac{n}{n}\right)-g\left(\tfrac{n-1}{n}\right)\right)}_{k=n}+\underbrace{f\left(\tfrac{2n+1}{n}\right)\cdot\left(g\left(\tfrac{2n+1}{n}\right)-g\left(\tfrac{2n}{n}\right)\right)}_{k=2n+1}\right)$$

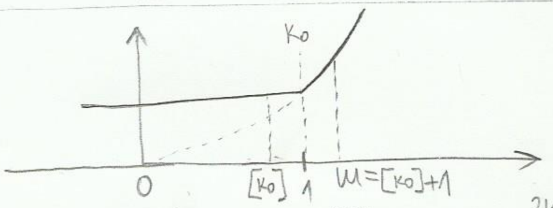
$$=\lim_{n\to\infty}\left(f(1)\cdot\left(g(1)-g\left(1-\tfrac{1}{n}\right)\right)+f\left(2+\tfrac{1}{n}\right)\left(g\left(2+\tfrac{1}{n}\right)-g(2)\right)\right)$$

$$=\lim_{n\to\infty}\left(f(1)\cdot(1-0)+f\left(2+\tfrac{1}{n}\right)\cdot(0-1)\right)=$$

$$=f(1)-\lim_{n\to\infty}f\left(2+\tfrac{1}{n}\right)=f(1)-f(2).$$

(2) $g(x)=\begin{cases}1, & x<1\\ x^2, & x\geq 1\end{cases}$

$$\int_{-5}^{5}x\,d\left(g(x)+x^2\right)=\int_{-5}^{5}x\,dg(x)+\int_{-5}^{5}x\,d(x^2)$$



за $-5+\frac{10}{n}\cdot k_0=1 \Rightarrow \frac{10k_0}{n}=6 \Rightarrow k_0=\frac{3n}{5}$

$m=[k_0]+1$

$$\to \int_{-5}^{5}x\,d(x^2)=\int_{-5}^{5}x\cdot 2x\,dx=\frac{2}{3}x^3\Big|_{-5}^{5}=\frac{500}{3}$$

$$\to \int_{-5}^{5}x\,dg(x)=\lim_{n\to\infty}\sum_{k=1}^{n}x_k\cdot\left(g(x_k)-g(x_{k-1})\right)=\lim_{n\to\infty}\sum_{k=m}^{n}\left(-5+\tfrac{10}{n}\cdot k\right)\cdot\left(g\left(-5+\tfrac{10}{n}\cdot k\right)-g\left(-5+\tfrac{10}{n}\cdot(k-1)\right)\right)=$$

$$P=\left\{-5,\ -5+\frac{10}{n},\ -5+\frac{10}{n}\cdot 2,\ \ldots,\ -5+\frac{10}{n}\cdot n\right\}$$

$$=\lim_{n\to\infty}\left(\underbrace{\left(-5+\frac{10}{n}\cdot m\right)\cdot\left(\left(-5+\frac{10}{n}\cdot m\right)^2-1\right)}_{k=m}+\sum_{k=m+1}^{n}\left(-5+\frac{10}{n}\cdot k\right)\cdot\left(\left(-5+\frac{10}{n}k\right)^2-\left(-5+\frac{10}{n}(k-1)\right)^2\right)\right)$$

$$=\lim_{n\to\infty}\sum_{k=m+1}^{n}\left(-5+\frac{10}{n}\cdot k\right)\cdot\left(\left(-5+\frac{10}{n}k\right)^2-\left(-5+\frac{10}{n}(k-1)\right)^2\right)=$$

$$=\lim_{n\to\infty}\int_{-5+\frac{10}{n}\cdot m}^{5}x\,d(x^2)=\lim_{n\to\infty}\int_{-5+\frac{10}{n}\cdot m}^{5}2x^2\,dx=\lim_{n\to\infty}\left(\frac{2}{3}x^3\right)\Bigg|_{-5+\frac{10}{n}\cdot m}^{5}=$$

$$=\lim_{n\to\infty}\left(\frac{250}{3}-\frac{2}{3}\left(-5+\frac{10}{n}\cdot m\right)^3\right)=\frac{250}{3}-\frac{2}{3}\cdot\left(-5+10\cdot\frac{3}{5}\right)^3=\frac{250}{3}-\frac{2}{3}\cdot 1=\frac{248}{3}.$$

$$\frac{\frac{3}{5}n}{n}\le\frac{m}{n}<\frac{\frac{3}{5}n+1}{n}\quad\Rightarrow\quad\lim_{n\to\infty}\frac{m}{n}=\frac{3}{5}$$

$$\frac{m}{n}=\frac{\left[\frac{3}{5}n\right]+1}{n}$$