ОДРЕЂЕНИ ИНТЕГРАЛ
-НАСТАВАК-

1 Задатак

Израчунати $I_n = \int_0^{\pi/2} \sin^n x\, dx$.

Решење

$$I_n = \int_0^{\pi/2} \sin^n x\, dx = \begin{bmatrix} u = \sin^{n-1} x & dv = \sin x\, dx \\ du = (n-1)\sin^{n-2} x \cdot \cos x\, dx & dv = d(-\cos x) \\ & v = -\cos x \end{bmatrix} =$$

$$= \sin^{n-1} x \cdot (-\cos x)\Big|_0^{\pi/2} - \int_0^{\pi/2} (-\cos x)\cdot(n-1)\sin^{n-2} x \cos x\, dx =$$

$$= -\sin^{n-1}\frac{\pi}{2}\cdot\underbrace{\cos\frac{\pi}{2}}_{0} + \underbrace{\sin^{n-1} 0}_{0}\cdot\cos 0 + (n-1)\int_0^{\pi/2} \sin^{(n-2)} x \cdot \cos^2 x\, dx =$$

$$= (n-1)\int_0^{\pi/2} \sin^{(n-2)} x \cdot (1-\sin^2 x)\, dx = (n-1)\left(\int_0^{\pi/2} \sin^{(n-2)} x\, dx - \int_0^{\pi/2} \sin^n x\, dx\right) =$$

$$= (n-1)\int_0^{\pi/2} \sin^{(n-2)} x\, dx - (n-1)\cdot\int_0^{\pi/2} \sin^n x\, dx =$$

$$= (n-1)I_{n-2} - (n-1)I_n$$

Одатле, имамо $I_n = (n-1)I_{n-2} - (n-1)I_n$

$$I_n + (n-1)I_n = (n-1)I_{n-2}$$

$$nI_n = (n-1)I_{n-2}$$

$$\boxed{I_n = \frac{n-1}{n} I_{n-2}}$$

Даље, $I_n = \frac{n-1}{n} I_{n-2} = \frac{n-1}{n}\cdot\frac{n-3}{n-2} I_{n-4}$

$$I_{2k} = \frac{(2k-1)!!}{(2k)!!} I_0 \quad , \quad I_{2k+1} = \frac{(2k)!!}{(2k+1)!!} I_1$$

$$I_0 = \int_0^{\pi/2} \sin^0 x\, dx = \int_0^{\pi/2} dx = x\Big|_0^{\pi/2} = \frac{\pi}{2} \;,\; I_1 = \int_0^{\pi/2} \sin^1 x\, dx = (-\cos x)\Big|_0^{\pi/2} = 1$$

$$\boxed{I_{2k}=\frac{(2k-1)!!}{(2k)!!}\cdot\frac{\pi}{2}\ ,\quad I_{2k+1}=\frac{(2k)!!}{(2k+1)!!}}$$

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2 Задатак

Израчунати $\int_0^{\pi} e^x\cos^2 x\,dx$

Решење:

$$I=\int_0^{\pi} e^x\cos^2 x\,dx=\int_0^{\pi} e^x\cdot\frac{1+\cos 2x}{2}dx=\frac{1}{2}\underbrace{\int_0^{\pi} e^x dx}_{I_1}+\frac{1}{2}\underbrace{\int_0^{\pi} e^x\cos 2x\,dx}_{I_2}=$$

$$I_1=\int_0^{\pi} e^x dx=e^x\Big|_0^{\pi}=e^{\pi}-e^0=e^{\pi}-1$$

$$I_2=\int_0^{\pi} e^x\cdot\cos 2x\,dx=*$$

Посматрајмо следеће:

$$\left.\begin{aligned}e^{i2x}&=\cos 2x+i\sin 2x\\ e^{(-2x)i}&=\cos(-2x)+i\sin(-2x)\end{aligned}\right\}+$$

Добијено $e^{2xi}+e^{-2xi}=2\cos 2x$

$$\boxed{\cos 2x=\frac{e^{2xi}+e^{-2xi}}{2}}$$

Заменимо то у *

$$I_2=\int_0^{\pi} e^x\cdot\left(\frac{e^{2xi}+e^{-2xi}}{2}\right)dx=\frac{1}{2}\int_0^{\pi} e^{(2i+1)x}dx+\frac{1}{2}\int_0^{\pi} e^{(-2i+1)x}dx=$$

$$=\frac{1}{2}\cdot\frac{1}{2i+1}\cdot e^{(2i+1)x}\Big|_0^{\pi}+\frac{1}{2}\cdot\frac{1}{-2i+1}e^{(-2i+1)x}\Big|_0^{\pi}=$$

$$=\frac{1}{4i+2}\cdot e^{(2i+1)\cdot\pi}-\frac{1}{4i+2}e^{(2i+1)0}+\frac{1}{-4i+2}e^{(-2i+1)\pi}-\frac{1}{-4i+2}e^{(-2i+1)0}=$$

Сада, посматрајмо $e^{(2i+1)\pi}=e^{\pi}e^{2\pi i}=e^{\pi}(\underset{1}{\cos 2\pi}+i\underset{0}{\sin 2\pi})=e^{\pi}$

$e^{\pi(1-2i)}=e^{\pi}e^{-2\pi i}=e^{\pi}\cdot(\cos(-2\pi)+i\sin(-2\pi))=e^{\pi}$

$$=\frac{1}{2}\left(\frac{1}{2i+1}e^{\pi}-\frac{1}{2i+1}+\frac{1}{-2i+1}e^{\pi}-\frac{1}{-2i+1}\right)=\frac{1}{2}(e^{\pi}-1)\left(\frac{1}{1+2i}+\frac{1}{1-2i}\right)$$

$=\frac{1}{2}(e^{\pi}-1)\cdot\frac{2}{5}=\frac{1}{5}(e^{\pi}-1)$, па је $I=\frac{1}{2}(e^{\pi}-1)+\frac{1}{10}(e^{\pi}-1)=\boxed{\frac{3}{5}(e^{\pi}-1)}$

3. Задатак

Нека је ф-ја f непрекидна

Доказати.

а) $\int_0^{\pi/2} f(\sin x)dx = \int_0^{\pi/2} f(\cos x)dx$

б) $\int_0^{\pi} x f(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$

в) Израчунати $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$

Решење

$$\int_0^{\pi/2} f(\sin x)dx = \left\{ \begin{array}{l} t=\frac{\pi}{2}-x \\ dt=-dx \end{array} \quad \begin{array}{c|c|c} x & 0 & \pi/2 \\ \hline t & \pi/2 & 0 \end{array} \right\} = \int_{\pi/2}^{0} f\left(\sin\left(\frac{\pi}{2}-t\right)\right)(-dt) =$$

$$= -\int_{\pi/2}^{0} f(\cos t)dt = \int_0^{\pi/2} f(\cos t)dt$$

$\boxed{\text{Искористили смо својство } a<b: \int_a^b f(x)dx = -\int_b^a f(x)dx}$

$$\text{б)} \int_0^{\pi} x f(\sin x)dx = \left\{ \begin{array}{l} t=\pi-x \\ dt=-dx \end{array} \quad \begin{array}{c|c|c} x & 0 & \pi \\ \hline t & \pi & 0 \end{array} \right\} = \int_{\pi}^{0} (\pi-t) f(\sin(\pi-t))(-dt) =$$

$$= \int_0^{\pi} (\pi-t) f(\sin t)dt = \int_0^{\pi} \pi f(\sin t)dt - \int_0^{\pi} t f(\sin t)dt \quad *$$

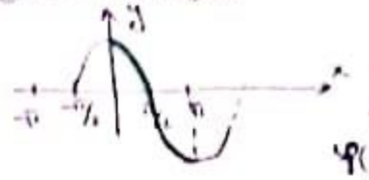
Запажање. Интеграл $\int_0^{\pi} x f(\sin x)dx$ је исти као $\int_0^{\pi} t f(\sin t)dt$ све је исто само је ознака променљиве различита и то не утиче на вредност интеграла, јер можемо искористити смену променљиве, нпр $t=x$ и проверити.

Означимо. $I = \int_0^{\pi} x f(\sin x)dx$ тада у $*$

$$I = \pi\int_0^{\pi} f(\sin x)dx - I \Rightarrow 2I = \pi\int_0^{\pi} f(\sin x)dx \Rightarrow \boxed{I = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx}$$

б) $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx = \int_0^{\pi} \frac{x\sin x}{1+1-\sin^2 x}dx = \underbrace{\frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+1-\sin^2 x}dx}_{\text{део под б)}} = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx =$

$$= \left\{ \begin{array}{l} t=\cos x \\ dt=-\sin x\,dx \end{array} \quad \begin{array}{c|c|c} x & 0 & \pi \\ \hline t & 1 & -1 \end{array} \right\} = \frac{\pi}{2}\int_1^{-1} \frac{-dt}{1+t^2} = -\frac{\pi}{2}\int_1^{-1}\frac{dt}{1+t^2} = \frac{\pi}{2}\int_{-1}^{1}\frac{dt}{1+t^2}$$



на интервалу $[0,\pi]$ ф-ја $\varphi(x)=\cos x$ је монотона

$$= \left\{ \begin{array}{l} \text{Сада имамо парну подинтегралну} \\ \text{ф-ју на симетричном домену} \end{array} \right\} = \frac{\pi}{2}\cdot 2\int_0^1 \frac{dt}{1+t^2} =$$

$$= \pi \operatorname{arctg} t\Big|_0^1 = \pi\cdot(\operatorname{arctg}1 - \operatorname{arctg}0) = \pi\cdot\left(\frac{\pi}{4}-0\right) = \frac{\pi^2}{4} \quad \blacksquare$$

4 Задатак

Израчунати: а) $\displaystyle\lim_{x\to 0^+} \frac{\int_0^x \cos t^2\,dt}{x}$ б) $\displaystyle\lim_{x\to +\infty} \frac{\left(\int_0^x e^{t^2}dt\right)^2}{\int_0^x e^{2t^2}dt}$

Решење:

а) Уочимо $\displaystyle\int_0^x \cos t^2\,dt \to 0$, када $x\to 0$

Присетимо се и следећег:
$$\left.\begin{array}{l} f:[a,b]\to\mathbb{R} \\ f\in C[a,b] \\ F(x)=\int_0^x f(t)dt,\ F(x)\in C[a,b] \end{array}\right\} \Rightarrow F'(x)=f(x)$$

Када посматрамо $\displaystyle\lim_{x\to 0^+}\frac{\int_0^x \cos t^2 dt}{x} = \left|\ \frac{0}{0}\text{, па примењујемо Лопиталово правило}\ \right|$

$$= \lim_{x\to 0^+} \frac{\left(\int_0^x \cos t^2 dt\right)'}{x'} = \lim_{x\to 0^+}\frac{\cos x^2}{1} = 1$$

б) Уочимо $e^{t^2} \geq 1,\ t\in[0,x)$

$$\int_0^x e^{t^2}dt \geq \int_0^x 1\,dt = x, \text{ тј. } \int_0^x e^{t^2}dt \geq x$$

Када $x\to+\infty$ имамо да онда и $\displaystyle\int_0^x e^{t^2}dt \to +\infty$ а одатле и $\displaystyle\int_0^x e^{2t^2}dt \to +\infty,\ x\to+\infty$.

Означимо са $f(x)=\left(\int\limits_0^x e^{t^2}dt\right)^2$ и са $g(x)=\int\limits_0^x e^{2t^2}dt$

Посматрамо $\lim\limits_{x\to+\infty}\frac{f(x)}{g(x)} = \left|\frac{+\infty}{+\infty}\text{, применићемо Лопиталово правило}\right| =$

$$=\lim_{x\to+\infty}\frac{f'(x)}{g'(x)} = \begin{bmatrix} f(x)=\left(\int\limits_0^x e^{t^2}dt\right)^2 & g(x)=\int\limits_0^x e^{2t^2}dt \\ f'(x)=2\cdot\int\limits_0^x e^{t^2}dt\cdot e^{x^2} & g'(x)=e^{2x^2}\end{bmatrix} =$$

$$=\lim_{x\to+\infty}\frac{2e^{x^2}\int\limits_0^x e^{t^2}dt}{e^{2x^2}} = \lim_{x\to+\infty}\frac{2\int\limits_0^x e^{t^2}dt}{e^{x^2}} = \left|\frac{+\infty}{+\infty}\text{, Лопиталово правило}\right| =$$

$$=\lim_{x\to+\infty}\frac{2e^{x^2}}{e^{x^2}\cdot 2x} = \lim_{x\to+\infty}\frac{1}{x}=0$$

$\boxtimes$

5 Задатак

Израчунати $\int\limits_0^1\frac{dx}{(x+1)\sqrt{x^2+1}}$

Решење:

Уведимо Ојлерову смену: $\sqrt{x^2+1}=t-x$.

Други начин.

$$I=\int\limits_0^1\frac{dx}{(x+1)\sqrt{x^2+1}} = \left\{\begin{array}{l} tg\,t=x \\ t=arctg\,x \\ dx=\frac{1}{\cos^2 t}dt\end{array}\quad \begin{array}{c|c|c} x & 0 & 1 \\ \hline t & 0 & \pi/4\end{array}\right\} = \int\limits_0^{\pi/4}\frac{\frac{1}{\cos^2 t}dt}{(tg\,t+1)\cdot\sqrt{tg^2t+1}} =$$

$$=\int\limits_0^{\pi/4}\frac{\frac{1}{\cos^2 t}dt}{\left(\frac{\sin t}{\cos t}+1\right)\sqrt{\frac{\sin^2 t}{\cos^2 t}+1}} = \int\limits_0^{\pi/4}\frac{\frac{1}{\cos^2 t}dt}{\frac{1}{|\cos t|}\cdot\frac{\sin t+\cos t}{\cos t}} = \left\{\begin{array}{l}\text{Пошто је } t\in[0,\frac{\pi}{4}] \\ \text{на том интервалу} \\ \cos t>0\end{array}\right\} =$$

$$=\int\limits_0^{\pi/4}\frac{\frac{1}{\cos^2 t}dt}{\frac{\sin t+\cos t}{\cos^2 t}} = \int\limits_0^{\pi/4}\frac{1}{\sin t+\cos t}dt = \frac{1}{\sqrt2}\cdot\int\limits_0^{\pi/4}\frac{1}{\frac{1}{\sqrt2}\sin t+\frac{1}{\sqrt2}\cos t}dt =$$

$$=\frac{1}{\sqrt2}\int\limits_0^{\pi/4}\frac{1}{\sin\left(\frac{\pi}{4}+t\right)}dt = \begin{bmatrix} u=\frac{\pi}{4}+t & \begin{array}{c|c|c} t & 0 & \pi/4 \\ \hline u & \pi/4 & \pi/2\end{array} \\ du=dt & \end{bmatrix} = \frac{1}{\sqrt2}\int\limits_{\pi/4}^{\pi/2}\frac{1}{\sin u}du =$$

$$=\frac{1}{\sqrt2}\ln tg\frac{t}{2}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{1}{\sqrt2}\Big(\underbrace{\ln tg\frac{\pi}{4}}_{=0} - \ln tg\frac{\pi}{8}\Big) = \boxed{-\frac{1}{\sqrt2}\ln tg\frac{\pi}{8}}$$

Израчунајмо $tg\frac{\pi}{8}$.

Имамо $1 = tg\frac{\pi}{4} = tg\,2\cdot\frac{\pi}{8} = \frac{2tg\frac{\pi}{8}}{1-tg^2\frac{\pi}{8}}$ (1)

Нека је $tg\frac{\pi}{8} = a$, $0<a<1$

Тада из (1) имамо $1 = \frac{2a}{1-a^2}$, тј. $2a = 1-a^2$

$$a^2+2a-1=0$$
$$\Rightarrow a_{1,2} = \frac{-2\pm\sqrt{4+4}}{2} = -1\pm\sqrt{2}$$

$$tg\frac{\pi}{8} = \sqrt{2}-1$$

Интеграл је сада: $I = -\frac{1}{\sqrt{2}}\ln(\sqrt{2}-1) = \frac{1}{\sqrt{2}}\ln\frac{1}{\sqrt{2}-1} = \frac{1}{\sqrt{2}}\ln\frac{\sqrt{2}+1}{(\sqrt{2}-1)(\sqrt{2}+1)} = \frac{1}{\sqrt{2}}\ln(1+\sqrt{2})$

$\boxtimes$

6. Задатак

Израчунати $\int\limits_{-1}^{1}\frac{dx}{x^2-2x\cos\alpha+1}$, $0<\alpha<\pi$

Решење:

$$\int\limits_{-1}^{1}\frac{dx}{x^2-2x\cos\alpha+1} = \int\limits_{-1}^{1}\frac{dx}{(x-\cos\alpha)^2+\sin^2\alpha} = \left\{\begin{array}{l} t = x-\cos\alpha \\ dt = dx \\ \begin{array}{c|c|c} x & -1 & 1 \\ \hline t & -1-\cos\alpha & 1-\cos\alpha \end{array}\end{array}\right\} =$$

$$= \int\limits_{-1-\cos\alpha}^{1-\cos\alpha}\frac{dt}{t^2+\sin^2\alpha} = \int\limits_{-1-\cos\alpha}^{1-\cos\alpha}\frac{dt}{\sin^2\alpha\left(\frac{t^2}{\sin^2\alpha}+1\right)} = \int\limits_{-1-\cos\alpha}^{1-\cos\alpha}\frac{1}{\sin^2\alpha}\cdot\frac{1}{\left(\left(\frac{t}{\sin\alpha}\right)^2+1\right)}dt =$$

$$= \frac{1}{\sin^2\alpha}\int\limits_{-1-\cos\alpha}^{1-\cos\alpha}\frac{1}{\left(\frac{t}{\sin\alpha}\right)^2+1}\cdot\sin\alpha\cdot d\left(\frac{t}{\sin\alpha}\right) = \frac{1}{\sin\alpha}\int\limits_{-1-\cos\alpha}^{1-\cos\alpha}\frac{d\left(\frac{t}{\sin\alpha}\right)}{\left(\frac{t}{\sin\alpha}\right)^2+1} = \frac{1}{\sin\alpha}\operatorname{arctg}\left(\frac{t}{\sin\alpha}\right)\Bigg|_{-1-\cos\alpha}^{1-\cos\alpha} =$$

$$= \frac{1}{\sin\alpha}\left(\operatorname{arctg}\frac{1-\cos\alpha}{\sin\alpha} - \operatorname{arctg}\frac{-1-\cos\alpha}{\sin\alpha}\right) = \frac{1}{\sin\alpha}\cdot\Big(\underbrace{\operatorname{arctg}\frac{1-\cos\alpha}{\sin\alpha}}_{u} + \underbrace{\operatorname{arctg}\frac{1+\cos\alpha}{\sin\alpha}}_{v}\Big)$$

$$= \frac{1}{\sin\alpha}\cdot(u+v)$$

$u = \operatorname{arctg}\frac{1-\cos\alpha}{\sin\alpha} \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, $1-\cos\alpha>0$, $\alpha\in(0,\pi) \Rightarrow \sin\alpha>0 \Rightarrow u\in\left(0,\frac{\pi}{2}\right)$

Аналогно $v\in\left(0,\frac{\pi}{2}\right)$

$tg\,u = \frac{1-\cos\alpha}{\sin\alpha}$, $tg\,v = \frac{1+\cos\alpha}{\sin\alpha}$, $tg\,u\cdot tg\,v = 1$

$\frac{\sin u}{\cos u}\cdot\frac{\sin v}{\cos v} = 1 \Rightarrow \sin u\cdot\sin v = \cos u\cos v$

$$\left.\begin{array}{l}\cos(u+v)=0 \\ 0<u+v<\pi\end{array}\right\} \Rightarrow \boxed{u+v=\frac{\pi}{2}}, \text{ тада } \boxed{I = \frac{1}{\sin\alpha}\cdot\frac{\pi}{2}}$$

$\boxed{7\ \text{Задатак}}$

Нека је $f(x) = x\left(\frac{\pi}{2} - \operatorname{arctg}\frac{1}{x}\right)$ и нека f има непрекидну инверзну ф-ју, израчунати $\int_0^{\pi/4} f^{-1}(y)\,dy$.

$\boxed{\text{Решење:}}$

Покажимо да ф-ја $f(x)$ има отклоњив прекид у $x=0$:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x\left(\frac{\pi}{2} - \operatorname{arctg}\frac{1}{x}\right) = \lim_{x\to 0^+} \underset{\downarrow\, 0^+}{x}\cdot\frac{\pi}{2} - \underset{\downarrow\, 0^+}{x}\,\underset{\downarrow\, \frac{\pi}{2}}{\operatorname{arctg}\frac{1}{x}} = 0$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x\left(\frac{\pi}{2} - \operatorname{arctg}\frac{1}{x}\right) = 0 \;,\; \operatorname{arctg}\frac{1}{x} \to -\frac{\pi}{2},\ x\to 0^-$$

Ф-ја $f(x)$ није дефинисана за $x=0$, а имамо $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^-} f(x)$ тако да имамо отклоњив прекид и можемо додефинисати ф-ју тако да буде непрекидна на $\mathbb{R}$

$$\tilde{f} = \begin{cases} x\left(\frac{\pi}{2} - \operatorname{arctg}\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases} \quad ,\ f'(x) = \frac{\pi}{2} - \operatorname{arctg}\frac{1}{x} + x\cdot\left(-\frac{1}{1+\left(\frac{1}{x}\right)^2}\cdot\left(-\frac{1}{x^2}\right)\right)$$

$$\int_0^{\pi/4} f^{-1}(y)\,dy = \left\{ \begin{array}{l} y = f(x) \\ dy = df(x) \\ dy = f'(x)\,dx \end{array} \quad \begin{array}{c|c|c} y & 0 & \pi/4 \\ \hline x & 0 & 1 \end{array} \right\} = \int_0^1 \underbrace{f^{-1}(f(x))}_{\text{идентичко пресликавање}} \cdot f'(x)\,dx =$$

$$= \int_0^1 x\cdot f'(x)\,dx = \int_0^1 x\cdot\left(\frac{\pi}{2} - \operatorname{arctg}\frac{1}{x} + x\cdot\frac{1}{1+\frac{1}{x^2}}\cdot\frac{1}{x^2}\right)dx =$$

$$= \int_0^1 x\cdot\frac{\pi}{2}\,dx - \underbrace{\int_0^1 x\cdot\operatorname{arctg}\frac{1}{x}\,dx}_{I_1} + \underbrace{\int_0^1 \frac{x^2}{x^2+1}\,dx}_{I_2} = \star$$

$$I_1 = \int_0^1 x\cdot\operatorname{arctg}\frac{1}{x}\,dx = \left[\begin{array}{ll} u = \operatorname{arctg}\frac{1}{x} & dv = x\,dx \\ du = \frac{1}{1+\frac{1}{x^2}}\left(-\frac{1}{x^2}\right)dx & dv = d\,\frac{x^2}{2} \\ & v = \frac{x^2}{2} \end{array} \right] =$$

$$= \operatorname{arctg}\frac{1}{x}\cdot\frac{x^2}{2}\Big|_0^1 - \int_0^1 \frac{x^2}{2}\cdot\left(-\frac{1}{1+x^2}\right)dx = \frac{1}{2}\operatorname{arctg}1 - 0 + \frac{1}{2}\underbrace{\int_0^1 \frac{x^2}{1+x^2}\,dx}_{I_2}$$

$$I_2 = \int_0^1 \frac{x^2}{1+x^2}\,dx = \int_0^1 \frac{x^2+1-1}{1+x^2}\,dx = \int_0^1 1\,dx - \int_0^1 \frac{1}{1+x^2}\,dx =$$

$$= x\Big|_0^1 - \operatorname{arctg}x\Big|_0^1 = 1 - 0 - (\operatorname{arctg}1 - \operatorname{arctg}0) = 1 - \frac{\pi}{4}$$

На крају, заменимо у $\star$ и добијамо $I = \frac{1}{2}$

Тада, даље имамо,

$$\star = \frac{\pi}{2}\cdot\frac{x^2}{2}\Big|_0^1 - \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2}\cdot\left(1-\frac{\pi}{4}\right) + \left(1-\frac{\pi}{4}\right) =$$

$$= \frac{\pi}{4} - \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} + 1 - \frac{\pi}{4} = \frac{1}{2}$$

На крају, закључујемо $\boxed{I = \frac{1}{2}}$

8. Задатак

Израчунати $\displaystyle\int_{\frac{1}{a}}^{a} \frac{\ln x}{1+x^2}\,dx$, $a>0$

Решење:

$$\int_{\frac{1}{a}}^{a} \frac{\ln x}{1+x^2}\,dx = \begin{bmatrix} u=\ln x & dv = \frac{1}{1+x^2}dx \\ du = d\ln x & dv = d\,\mathrm{arctg}\,x \\ du = \frac{1}{x}dx & v = \mathrm{arctg}\,x \end{bmatrix} =$$

$$= \ln x\cdot\mathrm{arctg}\,x\Big|_{\frac{1}{a}}^{a} - \underbrace{\int_{\frac{1}{a}}^{a}\mathrm{arctg}\,x\cdot\frac{1}{x}\,dx}_{I_1} = \star$$

$$I_1 = \int_{\frac{1}{a}}^{a}\mathrm{arctg}\,x\cdot\frac{1}{x}\,dx = \left\{ \begin{matrix} t = \frac{1}{x} & \begin{array}{c|c|c} x & \frac{1}{a} & a \\ \hline t & a & \frac{1}{a}\end{array} \\ dt = -\frac{1}{x^2}dx & \\ dt = -t^2dx & \end{matrix} \right\} =$$

$$= \int_{a}^{\frac{1}{a}}\mathrm{arctg}\frac{1}{t}\cdot t\cdot\left(-\frac{1}{t^2}\right)dt = -\int_{a}^{1/a}\frac{1}{t}\cdot\mathrm{arctg}\frac{1}{t}\,dt = \begin{bmatrix} u = \mathrm{arctg}\frac{1}{t} & dv = \frac{1}{t}dt \\ du = \frac{-1}{1+t^2}dt & v = \ln t \end{bmatrix}$$

$$= -\left(\mathrm{arctg}\frac{1}{t}\cdot\ln t\Big|_{a}^{\frac{1}{a}} - \int \ln t\cdot\frac{-1}{1+t^2}\,dt\right) =$$

$$= -\mathrm{arctg}\,a\cdot\ln\frac{1}{a} + \mathrm{arctg}\frac{1}{a}\cdot\ln a + \int_{\frac{1}{a}}^{a}\frac{\ln t}{1+t^2}\,dt$$

$$\star = \ln a\cdot\mathrm{arctg}\,a - \ln\frac{1}{a}\cdot\mathrm{arctg}\frac{1}{a} - \left(-\mathrm{arctg}\,a\cdot\ln\frac{1}{a} + \mathrm{arctg}\frac{1}{a}\cdot\ln a + I\right)$$

$$= \ln a\cdot\cancel{\mathrm{arctg}\,a} + \ln a\cdot\cancel{\mathrm{arctg}\frac{1}{a}} - \ln a\cdot\cancel{\mathrm{arctg}\,a} - \ln a\cdot\cancel{\mathrm{arctg}\frac{1}{a}} - I$$

На крају остане $I = -I \Rightarrow 2I = 0 \Rightarrow \boxed{I = 0}$

$\boxtimes$

8. Задатак

$\underset{\sim\sim\sim}{\text{II начин}}$

Израчунати $\int\limits_{\frac{1}{a}}^{a} \frac{\ln x}{1+x^2}dx$, $a>0$.

Решење.

$$I = \int\limits_{\frac{1}{a}}^{a} \frac{\ln x}{1+x^2}dx = \left[\begin{array}{ll} t = \frac{1}{x} & \begin{array}{c|c|c} x & \frac{1}{a} & a \\ \hline t & a & \frac{1}{a} \end{array} \\ dt = -\frac{1}{x^2}dx & \\ dt = -t^2dx & \end{array}\right] =$$

$$= \int\limits_{a}^{\frac{1}{a}} \frac{\ln\frac{1}{t}}{1+\frac{1}{t^2}}\cdot\left(-\frac{1}{t^2}\right)dt = -\int\limits_{a}^{\frac{1}{a}} \frac{-\ln t}{1+t^2}dt = \int\limits_{a}^{1/a} \frac{\ln t}{1+t^2}dt = -\int\limits_{\frac{1}{a}}^{a} \frac{\ln t}{1+t^2}dt$$

Сада, имамо:

$$I = -I \Rightarrow 2I = 0 \Rightarrow \boxed{I = 0}$$

9. Задатак

Реши интеграл $\int\limits_{0}^{2/3} \left(x - \frac{x^3}{3}\right)\cdot\frac{dx}{(1-x^2)\sqrt{1-2x^2}}$

Решење

$$I = \int\limits_{0}^{2/3} \left(x - \frac{x^3}{3}\right)\frac{dx}{(1-x^2)\sqrt{1-2x^2}} = \left\{\begin{array}{ll} t = 1-2x^2 & \\ x^2 = \frac{1-t}{2} & \begin{array}{c|c|c} x & 0 & \frac{2}{3} \\ \hline t & 1 & \frac{1}{9} \end{array} \\ dt = -4x\,dx & \end{array}\right\} =$$

$$= \int\limits_{0}^{2/3} \left(1 - \frac{x^2}{3}\right)\cdot\frac{x\,dx}{(1-x^2)\sqrt{1-2x^2}} = \int\limits_{1}^{1/9} \left(1 - \frac{\frac{1-t}{2}}{3}\right)\cdot\frac{-\frac{1}{4}dt}{\left(1-\frac{1-t}{2}\right)\cdot\sqrt{t}} =$$

$$= -\frac{1}{4}\int\limits_{1}^{1/9} \left(1 - \frac{1-t}{6}\right)\cdot\frac{dt}{\left(\frac{1+t}{2}\right)\cdot\sqrt{t}} = -\frac{1}{2}\int\limits_{1}^{1/9} \frac{5+t}{6}\cdot\frac{dt}{(1+t)\sqrt{t}} = \left\{\begin{array}{ll} u = \sqrt{t} & \begin{array}{c|c|c} t & 1 & \frac{1}{9} \\ \hline u & 1 & \frac{1}{3} \end{array} \\ t = u^2 & \\ dt = 2u\,du & \end{array}\right\} =$$

$$= -\frac{1}{12}\int\limits_{1}^{1/3} \frac{5+u^2}{(1+u^2)\cdot \cancel{u}}\cdot 2\cancel{u}\,du = -\frac{1}{6}\int\limits_{1}^{1/3} \frac{5+u^2}{1+u^2}du = -\frac{1}{6}\int\limits_{1}^{1/3} \frac{1+u^2+4}{1+u^2}du =$$

$$= -\frac{1}{6}\left(\int\limits_{1}^{1/3} du + 4\int\limits_{1}^{1/3} \frac{1}{1+u^2}du\right) = -\frac{1}{6}\left(u\Big|_1^{1/3} + 4\operatorname{arctg} u\Big|_1^{1/3}\right) =$$

$$= -\frac{1}{6}\left(\frac{1}{3} - 1 + 4\operatorname{arctg}\frac{1}{3} - 4\operatorname{arctg} 1\right) =$$

$$= -\frac{1}{6}\left(-\frac{2}{3} + 4\operatorname{arctg}\frac{1}{3} - \pi\right) = \boxed{\frac{1}{9} + \frac{\pi}{6} - \frac{2}{3}\operatorname{arctg}\frac{1}{3}}$$